On the alphabetical satisfiability problem for trace equations

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Abstract

In this paper we summarize some results of a work in progress on the computational complexity of the alphabetical satisfiability problem for linear trace equations. In particular we prove that the problem is in P under some conditions on the maximal I-cliques.

1 Introduction

In his seminal paper Makanin [7] gave a very complicated algorithm to decide whether or not a word equation with constants has a solution. Later several authors considered the problem of satisfiability of equations by a solution $\{\varphi(x) \mid x \in \Xi\}$ satisfying some constraints. In the second half of '90 attention was paid also to equations on free partially commutative monoids. Free partially commutative monoids, firstly introduced in combinatorics [2], became very important in computer science for the theory of concurrence in connection with the semantics of labelled Petri nets [9] and the investigation of parallel program schemata [6]. The decidability of the satisfiability problem for equations on free partially commutative monoids, trace equations for short, was proved by Matiyasevich in [8] and by Diekert and al. [3], [4].

We proved [1] that the uniform satisfiability problem for linear trace equations (i.e. for the problem where the independence alphabet (Σ, I) is considered as variable parameter) and the non uniform problem for quadratic trace equations are NP-complete problems even if we look for a solution whose elements belong to Σ. So the following problem is quite natural: to which complexity class belongs the (non uniform) alphabetical satisfiability problem for linear trace equations? In the extremal cases of free monoids and free commutative monoids the problem is obviously linear, in [1] we gave also a linear algorithm that solves the problem in the case of partially commutative monoids with disjoint I-cliques, and now we are considering other topologies of the independence relation graph to have some hints to answer the above question. In particular here we shortly describe a polynomial algorithm that solves the question when each pair of maximal I-cliques has the same intersection.

2 Preliminaries

Let Σ be a finite alphabet and let $I \subseteq \Sigma \times \Sigma$ be a binary irreflexive and symmetric relation, called *independence* relation. We denote by $\sim_I$ the least congruence over Σ^* generated by the relations $ab = ba$, for all $(a, b) \in I$. The pair (Σ, I) is called *independence alphabet*. For a subset A of Σ, let $I_A = (A \times A) \cap I$. If $I_A = A \times A$, then A is called a *clique* of the independence alphabet, or a I-clique.

The *free partially commutative* monoid (or trace monoid) over (Σ, I), is the quotient $M(\Sigma, I) = \Sigma^*/\sim_I$ and it can be also denoted by M, when no confusion arises. The elements of M are called *traces* and the trace with representative $x \in \Sigma^*$ is denoted by $[x]$.
Let Ξ be a finite set of unknowns and $\Theta = \Sigma \cup \Xi$. A *trace equation* with constants over (Σ, I) has the form $w_L \equiv w_R$ with $w_L, w_R \in \Theta^+$ such that $w_L w_R$ contains both unknowns from Ξ and constants from Σ. A trace equation $w_L \equiv w_R$ is called *linear* if each unknown occurs at most once in $W_L W_R$.
An *alphabetical assignment* is a map $\varphi : \Xi \to \Sigma$. It can be extended to the monoid homomorphism $\varphi^* : \Theta^* \to \Sigma^*$ by putting $\varphi(a) = a$ for all $a \in \Sigma$. We say that the trace equation $w_L \equiv w_R$ is *alphabetically satisfiable* if $\varphi^*(w_L) \sim_I \varphi^*(w_R)$ for some alphabetical assignment φ. In such case we say also that φ satisfies $w_L \equiv w_R$ and the set $\{\varphi(x) \mid x \in \Xi\}$ is called an *alphabetical solution* of $w_L \equiv w_R$.
Let $C_1, C_2, \ldots, C_k$ be the set of maximal I-cliques of (Σ, I) and assume that there is a subset Z of Σ such that $C_i \cap C_j = Z$ for all $1 < j \leq k$, in the sequel we call Z *center* of the trace monoid, each $z \in Z$ *central* element and each free partially commutative monoid on a independence alphabet that satisfies the above condition *central partially commutative monoid.*

3 Main result

We claim that *the alphabetical satisfiability problem for linear trace equation on a central partially commutative monoid is in P.*
To give an idea of the algorithm we need some notation.
Let $w \in \Theta^+$, we call *block* of w each factor of the form $\xi\pi$ with $\xi \in \Xi^*$, $\pi \in (C_i \setminus Z)^+$ for some $1 \leq i \leq k$. We say that a word $w \in \Theta^+$ is in *normal form* if

$$w = (\xi_1\pi_1)(\xi_2\pi_2)\ldots(\xi_r\pi_r)\xi_{r+1}\zeta\xi_{r+2}$$

where $r \geq 0$, $\zeta \in Z^*$, $\xi_{r+1}\xi_{r+2} \in \Xi^*$, and for each $i, 1 \leq i \leq r$ $\xi_i\pi_i$ is a block of maximal length. It is easy to check that for each w there is a unique $w' \in \Theta^+$ in normal form with $\xi_{r+2} = \epsilon$ such that for each alphabetical assignment ϕ $[\phi^*(w)] = [\phi^*(w')]$.
Then without loss of generality we can consider linear trace equations $w_1 \equiv w_2$ where $w_1 =, w_2$ are in normal form, since a linear trace equation $w_1 \equiv w_2$ is alphabetically satisfiable if and only if the equation $w_1' \equiv w_2'$ is alphabetically satisfiable. Let

$$w_1 = (\xi_{1,1}\pi_{1,1})(\xi_{1,2}\pi_{1,2})\ldots(\xi_{1,t}\pi_{1,t})\xi_{1,t+1}\zeta_1\xi_{1,t+2}$$

and

$$w_2 = (\xi_{2,1}\pi_{2,1})(\xi_{2,2}\pi_{2,2})\ldots(\xi_{2,s}\pi_{2,s})\xi_{2,s+1}\zeta_2\xi_{2,s+2}$$

(where we can assume $\xi_{1,t+2} = \xi_{2,s+2} = \epsilon$).
Assume $t > 0, s > 0$. To decide whether the equation $w_1 \equiv w_2$ is alphabetically satisfiable we start considering the first two blocks $(\xi_{1,1}\pi_{1,1})$, $(\xi_{2,1}\pi_{2,1})$ of w_1,w_2 respectively. We have to distinguish two different cases.
CASE 1. Let $\pi_{1,1} \in (C_i \setminus Z)^+$, $\pi_{2,1} \in C_j \setminus Z)^+$ with $i \neq j$, then if $|\pi_{1,1}| > |\xi_{2,1}| \wedge |\pi_{2,1}| > |\xi_{1,1}|$ (where $|x|$ denotes as usual the length of x) no assignment can satisfy the equation.
If $|\pi_{i,1}| \leq |\xi_{2,1}|$ we consider the words $u_1^{(1)} = (\xi_{1,2}\pi_{1,2})\ldots(\xi_{1,t}\pi_{1,t})\xi_{1,t+1}\zeta_1(\xi_{1,t+2}\xi_{1,1})$ and $u_2^{(1)} = (\xi_{2,1}'\pi_{2,1})(\xi_{2,2}\pi_{2,2})\ldots(\xi_{2,s}\pi_{2,s})\xi_{2,s+1}\zeta_2\xi_{2,s+2}$, where $\xi_{2,1}'$ is the suffix of $\xi_{2,1}$ of length $|\xi_{2,1}| - |\pi_{1,1}|$.

We remark that, for $i = 1,2$, $u_i^{(1)}$ is a word in normal form with $|u_i^{(1)}| < |w_i|$ and that $u_1^{(1)}$ has one block less than w_1. If $|\pi_{2,1}| \leq |\xi_{1,1}|$ we proceed exchanging the roles of w_1 and w_2 getting two words $u_1^{(2)}$,$u_2^{(2)}$ in normal form where $|u_i^{(2)}| < |w_i|$ and that $u_2^{(2)}$ has one block less than w_2. The linear trace equation $w_1 \equiv w_2$ is alphabetically satisfiable if and only if for some $i = 1,2$ the equation $u_1^{(i)} \equiv u_2^{(i)}$ does.
CASE 2. Let $\pi_{1,1}, \pi_{2,1} \in (C_j \setminus Z)^+$ for some j, $1 \leq j \leq k$. Let $\pi'_{1,1}, \pi'_{2,1}$ be the words obtained by deleting from $\pi_{1,1}, \pi_{2,1}$ for each $a \in C_j$ their common subword a^d of maximal length. Then if $|\pi'_{1,1}| > |\xi_{2,1}| + |\xi_{2,2}| \wedge |\pi'_{2,1}| > |\xi_{1,1}| + |\xi_{1,2}| \wedge \pi_{1,2} \notin (C_j)^+ \wedge \pi_{2,2} \notin (C_j)^+$ then no alphabetical assignment satisfies the equation. If $|\pi'_{1,1}| \leq |\xi_{2,1}| + |\xi_{2,2}|$ then consider the words $u_1^{(1)} = (\xi_{1,2}\pi_{1,2}) \ldots (\xi_{1,t}\pi_{1,t})\xi_{1,t+1}\zeta_1(\xi_{1,t+2}\xi_{1,1})$ and $u_2^{(1)} = (\xi'_{2,1}\pi_{2,1})(\xi'_{2,2}\pi_{2,2}) \ldots (\xi_{2,s}\pi_{2,s})\xi_{2,s+1}\zeta_2\xi_{2,s+2}$, where if $m = |\xi_{2,1}| - |\pi_{1,1}| \geq 0$ then $\xi'_{2,1}$ is the suffix of $\xi_{2,1}$ of length m and $\xi'_{2,2} = \xi_{2,2}$ otherwise, if $m < 0$, $\xi_{2,1} = \epsilon$, and $\xi'_{2,2}$ is the suffix of $\xi_{2,2}$ of length $|\xi_{2,2}| + m$. We remark again that for $i = 1,2$ $u_i^{(1)}$ is a word in normal form with $|u_i^{(1)}| < |w_i|$ and that $u_1^{(1)}$ has one block less than w_1. If $|\pi'_{2,1}| \leq |\xi_{1,1}| + |\xi_{1,2}|$ we proceed exchanging the roles of w_1 and w_2 getting two words $u_1^{(2)}$,$u_2^{(2)}$.
If $\pi_{1,2} \in (C_j \setminus Z)^+$ we consider the words $\widetilde{u_1}^{(1)} = (\xi_{1,1}\xi_{1,2}\pi'_{1,1}\pi_{1,2}) \ldots (\xi_{1,t}\pi_{1,t})\xi_{1,t+1}\zeta_1(\xi_{1,t+2}\xi_{1,1})$ and $\widetilde{u_2}^{(1)} = w_2$. Again for $i = 1,2$ $\widetilde{u_i}^{(1)}$ is a word in normal form with $|\widetilde{u_i}^{(1)}| \leq |w_i|$ and that $\widetilde{u_1}^{(1)}$ has one block less than w_1. If $\pi_{2,2} \in (C_j \setminus Z)^+$ we proceed exchanging the roles of w_1 and w_2 getting two words $\widetilde{u_1}^{(2)}$, $\widetilde{u_2}^{(2)}$ not longer than w_1, w_2 respectively where $\widetilde{u_2}^{(2)}$ has one block less than w_2 . The linear trace equation $w_1 \equiv w_2$ is alphabetically satisfiable if and only if for some $i = 1,2$ either the equation $u_1^{(i)} \equiv u_2^{(i)}$ or the equation $\widetilde{u_1}^{(i)} \equiv \widetilde{u_2}^{(i)}$ is so.
So either we get that the equation $w_1 \equiv w_2$ has no alphabetical solution or we reduce the alphabetical satisfiability problem of $w_1 \equiv w_2$ to the alphabetical satisfiability problem of a linear trace equation $u_1 \equiv u_2$ where u_1, u_2 are words not longer than w_1, w_2 in normal form and one of them, say u_i has less blocks than w_i.
Then if both u_1, u_2 have at least one block we can continue as before, otherwise we assume that at least one of them, say u_1, has no block. So let $u_1 = x_1\zeta_1 x2$ with $x_1, \xi_{1,2} \in \Xi^*$, $\zeta_1 \in Z^*$ and $u_2 = (\xi_{2,1}\pi_{2,1})(\xi_{2,2}\pi_{2,2}) \ldots (\xi_{2,q}\pi_{2,q})\xi_{2,q+1}\zeta_{22,q+2}$, with $q \geq 0$.
If $|\pi_{2,1}\pi_{2,2} \ldots \pi_{q,n}| > |x_1|$ the equation $u_1 \equiv u_2$ is not alphabetically satisfiable, otherwise we put $v_1 = x'_1\zeta_1 x_2$ where x'_1 is the suffix of x_1 of length $|x_1| - |\pi_{2,1}\pi_{2,2} \ldots \pi_{2,q}|$ and $v_2 = \zeta_2\xi_{2,q+2}\xi_{2,1} \ldots \xi_{2,q+1}$ and $u_1 \equiv u_2$ is alphabetically satisfiable if and only if $v_1 \equiv v_2$ is so.
Let z_1, z_2 be the words obtained by deleting from ζ_1 and ζ_2 their longest common subword in $\{c\}^*$ for each central element c. The equation $v_1 \equiv v_2$ is alphabetically satisfiable if and only if $|z_1| \leq |\xi_{2,q+2}\xi_{2,1}\xi_{2,2} \ldots \xi_{2,q+1}| \wedge |z_2| \leq |x'_1 x_2|$.
We roughly described an algorithm that solves the alphabetical satisfiability problem for linear trace equations on central partially commutative monoids. This algorithm looks for a solution that has at most unknowns as possible assigned in the cente. We can prove that the algorithm is in P by codifying its steps in a suitable automaton. A rough bound of its cost is $O(|w_1|^{8+|\Sigma|})$.

4 Conclusion

It is still an open problem to determine the complexity class of the general alphabetical satisfiability problem for a linear trace equation. We hope that a technique similar to the previous one can be used for other topologies like for instance transitive trees and forests and that these particular

cases can suggest the right way to deal with the problem.
Moreover some other problems similar to the ones considered for equations on the free monoid can be considered, like the complexity of the problem when the solution can assign the empty word to some unknown or when to the unknowns can be assigned words whose length is bounded by a given k.

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