The Strength of Parameterized Tree-like Resolution

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Abstract

We examine the proof-theoretic strength of parameterized tree-like resolution—a proof system for the coW[2]*-complete set of parameterized tautologies. Parameterized resolution and, moreover, a general framework for parameterized proof complexity was introduced by Dantchev, Martin, and Szeider (FOCS'07). In that paper, Dantchev et al. show a complexity gap in parameterized tree-like resolution for propositional formulas arising from translations of first-order principles.*

Here we pursue a purely combinatorial approach to obtain lower bounds to the proof size in parameterized tree-like resolution. For this we devise a prover-delayer game suitable for parameterized resolution. By exhibiting good delayer strategies we then show lower bounds for the pigeonhole principle as well as the order principle. On the other hand, we demonstrate that parameterized tree-like resolution is a very powerful system, as it allows short refutations of all parameterized contradictions given as bounded-width CNF's. Thus, a number of principles such as Tseitin tautologies, pebbling contradictions, or random 3-CNF's which serve as hard examples for classical resolution become easy in the parameterized setting.

1 Introduction

Proof complexity is a research field which owes one of its main motivations from the problem of separating complexity classes as P, NP, and coNP, using an approach which integrates techniques and results from mathematical logic, model theory, combinatorics, and computational complexity. Cook and Reckhow [2] initiated the study of lengths of proofs in propositional proof systems. Their result that the existence of a *polynomially bounded* propositional proof system, i. e., a proof system where all tautologies have polynomial-size proofs, is equivalent to $\mathsf{NP} = \mathsf{coNP}$, has opened the way to proving lower bounds for the lengths of proofs in a diversity of propositional proof systems ranging from restricted versions of resolution to bounded-depth

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Frege systems (see [6] for a recent survey on the field). While all these systems are known to be *not* polynomially bounded, still a lot of effort has to be invested to reach, for instance, super-polynomial lower bounds for Frege systems.

Recently, Dantchev, Martin, and Szeider [3] introduced and initiated the study of *parameterized proof complexity*. After considering the notions of propositional *parameterized tautologies* and *fpt-bounded* proof systems, they laid the foundations to study complexity of proofs in a parameterized setting. The main motivation behind their work was that of generalizing the classical approach of Cook and Reckhow to the parameterized case and working towards a separation of parameterized complexity classes as FPT and W[P] by techniques developed in proof complexity.

In this work we continue the study of the complexity of proofs in parameterized tree-like resolution. As our main contribution (Section 3) we devise a purely combinatorial approach, based on a prover-delayer game, to characterize proof size in parameterized tree-like resolution. In particular, we use our characterization to prove lower bounds. Our game is inspired by the prover-delayer game of Pudlák and Impagliazzo [5], which is one of the canonical tools to study lower bounds in tree-like resolution [1,5] and tree-like $Res(k)$ [4]. In fact, we show that the game from [5] is a very simple case of our game.

2 Preliminaries

Definition 2.1 (Dantchev, Martin, Szeider [3]). *A* parameterized contradiction *is a pair* (F, k) *consisting of a propositional formula* F *and* $k \in \mathbb{N}$ *such that* F *has no satisfying assignment of weight* $\leq k$. *We denote the set of all parameterized contradictions by* PCon.

A *literal* is a positive or negated propositional variable and a *clause* is a set of literals. A clause is interpreted as the disjunctions of its literals and a set of clauses as the conjunction of the clauses. Hence clause sets correspond to formulas in CNF. The *parameterized resolution system* [3] is a refutation system for the set of all parameterized contradictions. Any line in the proof is either (I) a clause of F; (II) a parameterized axiom of the form $\neg x_{i_1} \vee \cdots \vee \neg x_{i_{k+1}}$ for any set of $k+1$ variables $\{x_{i_1}, \ldots, x_{i_{k+1}}\}$; or (III) a clause obtained by the *resolution rule*

$$\frac{\{x\} \cup C \qquad \{\neg x\} \cup D}{C \cup D}$$

for previous proof lines $C \vee x, D \vee \neg x$. The aim of a resolution refutation is to demonstrate unsatisfiability of a clause set by deriving the empty clause. We focus on refutations where every derived clause is used at most once as a prerequisite of the resolution rule. Such refutations are called *tree-like*. The *size* of a resolution refutation is the number of its lines. Notice that any tree-like resolution refutation is essentially a boolean decision tree which computes the axiom or the initial clause by an assignment. This interpretation helps to understand our lower bounds.

3 Lower Bounds via a Prover-Delayer Game

Let $(F, k) \in \mathrm{PCon}$ where F is a set of clauses in n variables $x_1, \ldots, x_n$. We define a prover-delayer game: prover and delayer build a (partial) assignment to $x_1, \ldots, x_n$. The game is over as soon as the partial assignment falsifies either a clause from F or a parameterized clause

$\neg x_{i_1} \vee \cdots \vee \neg x_{i_{k+1}}$ where $1 \leq i_1 < \cdots < i_{k+1} \leq n$. The game proceeds in rounds. In each round, prover suggests a variable x_i, and delayer either chooses a value 0 or 1 for x_i or leaves the choice to the prover. In this last case, if the prover sets the value, then the delayer gets some points. The number of points delayer earns depends on the variable x_i, the assignment α constructed so far in the game, and two functions $c_0(x_i, \alpha)$ and $c_1(x_i, \alpha)$. More precisely, the number of points that delayer will get is

$$\begin{array}{cl} 0 & \text{if delayer chooses the value,} \\ \log c_0(x_i, \alpha) & \text{if prover sets } x_i \text{ to 0, and} \\ \log c_1(x_i, \alpha) & \text{if prover sets } x_i \text{ to } 1. \end{array}$$

Moreover, the functions $c_0(x, \alpha)$ and $c_1(x, \alpha)$ are chosen in such a way that for each variable x and assignment α

$$\frac{1}{c_0(x, \alpha)} + \frac{1}{c_1(x, \alpha)} = 1 \tag{1}$$

holds. Let us call this game the (c_0, c_1)-game on (F, k).

The connection of this game to size of proofs in parameterized tree-like resolution is given by the next theorem:

Theorem 3.1. *Let (F, k) be a parameterized contradiction and let c_0 and c_1 be two functions satisfying (1) for all partial assignments α to the variables of F. If (F, k) has a tree-like parameterized resolution refutation of size at most S, then the delayer gets at most $\log S$ points in each (c_0, c_1)-game played on (F, k).*

By setting $c_0(x, \alpha) = c_1(x, \alpha) = 2$ for all variables x and partial assignments α, we get the game of Pudlák and Impagliazzo [5]. Suitably choosing functions c_0 and c_1 and defining a delayer-strategy for the (c_0, c_1)-game we can prove a lower bound to the proof size in tree-like parameterized resolution. We will illustrate this for the *pigeonhole principle* PHP^{n+1}_n which uses variables $x_{i,j}$ with $i \in [n+1]$ and $j \in [n]$, indicating that pigeon i goes into hole j. PHP^{n+1}_n consists of the clauses

$$\bigvee_{j \in [n]} x_{i,j} \quad \text{for all pigeons } i \in [n+1] \text{ and } \quad \neg x_{i_1,j} \vee \neg x_{i_2,j}$$

for all choices of distinct pigeons $i_1, i_2 \in [n+1]$ and holes $j \in [n]$. The following theorem shows that PHP^{n+1}_n is hard for parameterized tree-like resolution.

Theorem 3.2. *PHP^{n+1}_n has no fpt-size parameterized tree-like resolution refutation.*

By inspection of the above delayer strategy it becomes clear that the lower bound from Theorem 3.2 also holds for the *functional pigeonhole principle* where in addition to the clauses from PHP^{n+1}_n we also include $\neg x_{i,j_1} \vee \neg x_{i,j_2}$ for all pigeons $i \in [n+1]$ and distinct holes $j_1, j_2 \in [n]$.

4 Kernels and Small Refutations

The notion of *efficient kernelization* plays an important role in the theory of parameterized complexity. A set $\Gamma \subseteq \mathrm{PCon}$ of parameterized contradictions has a *kernel* if there exists a computable function f such that every $(F, k) \in \Gamma$ has a subset $F' \subseteq F$ of clauses satisfying the following conditions: (1) F' contains at most $f(k)$ variables and (2) (F', k) is a parameterized contradiction. Some examples of CNF's with kernels are: **Pebbling contradictions**, **Non c-Colorability**, **Graph pigeonhole principle**, some encoding of **Linear Ordering principles**, any unsatisfiable **Bounded-width CNF**. The kernels in the previous examples are very explicit, but this is not always the case. Is it easy to find a kernel if it is known to exist? The answer to this question has consequences regarding automatizability of tree-like parameterized resolution. We show a general strategy for finding kernels and fpt-bounded refutations for parameterized contradictions of bounded width, which are usually hard for non-parameterized proof complexity.

Theorem 4.1. *If F is a CNF of width w and (F, k) is a parameterized contradiction, then (F, k) has a parameterized tree-like resolution refutation of size $O(w^{k+1})$. Moreover, there is an algorithm that for any (F, k) either finds such tree-like refutation or finds a satisfying assignment for F of weight $\leq k$. The algorithm runs in time $O(|F| \cdot k \cdot w^{k+1})$.*

5 Ordering Principles

In this section we discuss parameterized resolution refutations for various *ordering principles*. The principle claims that any finite partially ordered set has a minimal element. We are also interested in the *linear ordering principle* in which the set is required to be *totally* ordered.

Theorem 5.1. *The linear ordering principle has fpt-bounded tree-like refutations in parameterized resolution.*

The following theorem has been first proved in [3]. Their proof is based on a model-theoretic criterion. A combinatorial proof exists which is based on prover-delayer games.

Theorem 5.2. *Ordering Principle has no fpt-bounded tree-like parameterized resolution refutations.*

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