Is there something more to say about L(2,1)-coloring of graphs?

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The problem

The L(2,1)-labeling problem consists of assigning colors from the integer set 0…λ to the vertices of a graph G s.t.:

- vertices at distance two have colors which differ by at least 1
- adjacent vertices have colors which differ by at least 2

the aim is to minimize λ

Mobile computing

The L(2,1)-labeling problem was introduced by Griggs and Yeh in 1992 in relation to multihop radio network (indeed this problem was before mentioned by Roberts in 1991 in a survey on colorings).

The task is to assign radio frequencies to the trasmitters at different locations without causing interference.

the aim is to minimize the frequency bandwidth

…therefore

The situation is modeled by a graph

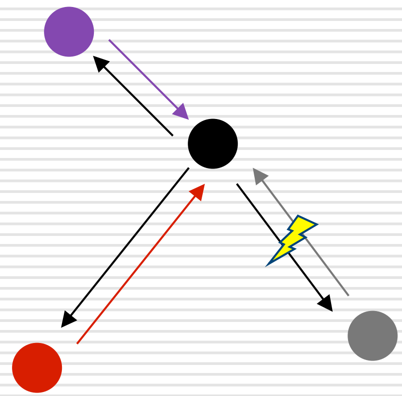
vertices	radio trasmitters/receivers
edges	possible communications (hence interferences)
λ	frequency bandwidth

To avoid frequencies collisions

direct collisions

(the frequencies of a station and its neighbors must be sufficiently different that their signals will not interfere)

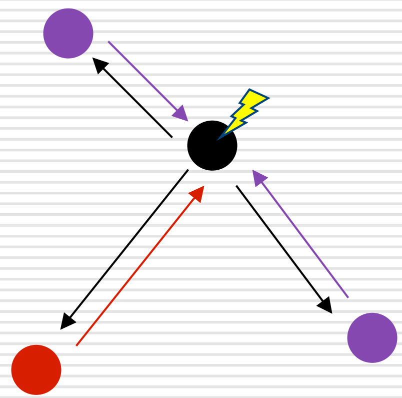
adjacent vertices get colors which are at least 2 apart



hidden collisions

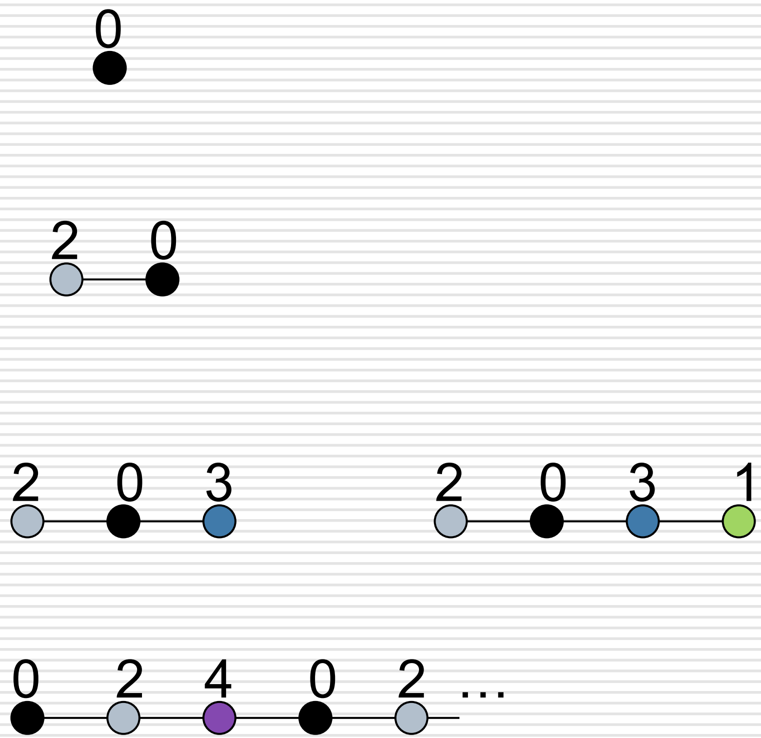
(a station must not receive signals of the same frequency from any of its neighbors)

vertices at distance of at most two get colors which are at least 1 apart



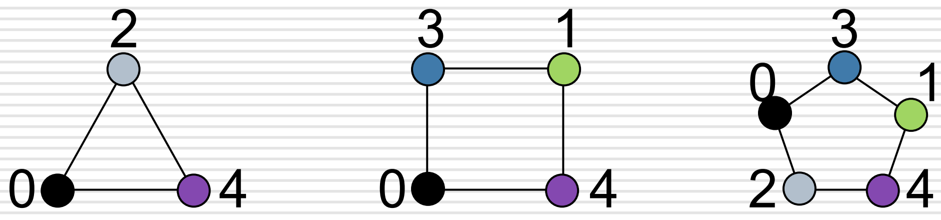
L(2,1)-labeling paths

$$\lambda_{2,1}(P_n) = \begin{cases} 0 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \text{ or } 4 \\ 4 & \text{if } n >= 5 \end{cases}$$

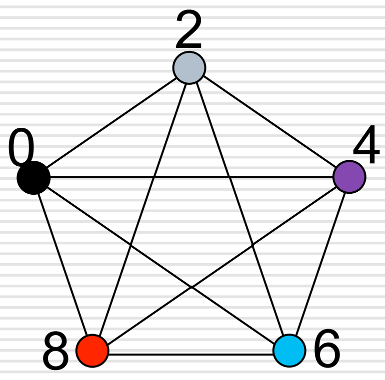


L(2,1)-labeling cycles and cliques

$\lambda_{2,1}(C_n) = 4$ *if n >= 3*



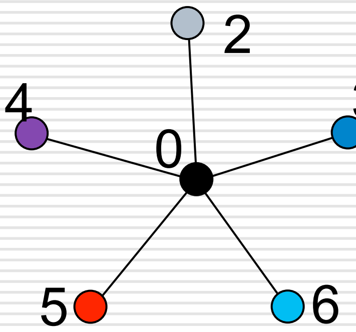
$\lambda_{2,1}(K_n) = 2(n-1)$



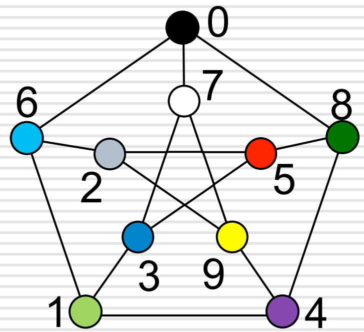
Lower and upper bounds for L(2,1)-labelling

Conjecture: $\lambda_{2,1}(G) <= \Delta^2$ *for any graph*

$\lambda_{2,1}(G) >= \Delta + 1$



$\lambda_{2,1}(G) = \Delta^2$



*Petersen**

* *Moore graphs: Regular graphs with given diameter, maximum degree and minimum number of nodes (for n=10 K_{10} , $K_{5,5}$)*

Complexity

L(2,1) is NP-Complete (Griggs,Yeh 92)

Reduction from hamiltonian path

Restriction to particular classes of graphs

……. to understand

when L(2,1) remains NP-Complete

when L(2,1) is polynomial

when it is possible to find upper and/or lower bounds

L(2,1)-labeling remains NP-complete

L(2,1)-labeling remains NP-complete also when restricted to:

Bipartite, Split and Chordal graphs (Boadlander, Kloks, Tan, van Leeuwen 2004)

d-regular graphs (Fiala, Kratochvil 2005)

Planar graphs (Eggemann,Havet,Noble 2009)

…………

L(2,1)-labeling is polynomial ......

- Hypercubes (Whittlesey,Georges,Mauro 1995)
- Cographs (Chang, Kuo 1996)
- Regular Grids (Calamoneri, Petreschi 2004)
- Trees (Hasunuma,Ishii,Ono,Uno 2009)

....

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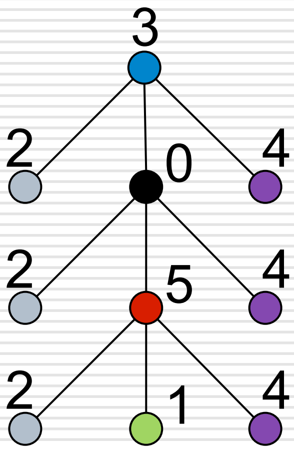
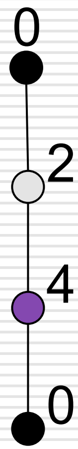
L(2,1)-labeling trees



$\lambda_{2,1}(T) = \Delta + 1$



$\lambda_{2,1}(T) = \Delta + 2$

1992 Griggs and Yeh conjecture that the recognition of these classes is NP-hard

1996 Chang and Kuo disprove the conjecture by presenting a polynomial algorithm

2008 Hasunuma, Ishii, Ono and Uno present a linear time algorithm

Upper bounds

- Chordal graphs

$\lambda_{2,1}(G) <= 1/4(\Delta +3)^2$ (Sakai 1994)

- Graphs of treewidth bounded by t

$\lambda_{2,1}(G) <= t\Delta +2t$ (Boadlander, Kloks, Tan, van Leeuwen 2004)

- outerplanar graphs with maximum degree $\Delta >=8$

$\lambda_{2,1}(G) <= \Delta +2$ (Calamoneri,Petreschi 2004)

- Unit disk graphs

$\lambda_{2,1}(G) <= 4/5\Delta^2 +2\Delta$ (Shao, Yeh, Shiu 2008)

- Co-comparability graphs

$\lambda_{2,1}(G) <= 8\Delta +1$ (Calamoneri,Caminiti,Olariu,Petreschi 2008)

……………………………………

Open problems on L(2,1)

- find other classes for which L(2,1) remains NP-complete
- find other classes for which L(2,1) is polynomial
- find other classes for which it is possible to compute upper and/or lower bounds
- Complete the proof that $\lambda_{2,1}(G) <= \Delta^2$ for any graph

Since L(2,1)-labeling is really an intriguing problem.....

the conjecture $\lambda_{2,1}(G) \le \Delta^2$ for any graph

1992 Griggs,Yeh	$\lambda_{2,1}(G) \le \Delta^2 + 2\Delta$
1993 Jonas	$\lambda_{2,1}(G) \le \Delta^2 + 2\Delta - 4$
1996 Chang,Kuo	$\lambda_{2,1}(G) \le \Delta^2 + \Delta$
2003 Kral,Skrekovski	$\lambda_{2,1}(G) \le \Delta^2 + \Delta - 1$
2008 Goncalves	$\lambda_{2,1}(G) \le \Delta^2 + \Delta - 2$
2008 Havet, Reed,Sereni	$\lambda_{2,1}(G) \le \Delta^2$ $\Delta >= 10^{69}$

… interest for generaliztions is grown up

- L(h,k)
- oriented graphs

Basic L(h,k)-labeling problem

The L(h,k)-labeling problem consists of assigning colors from the integer set 0…λ to the vertices of a graph G s.t.:

- vertices at distance two have colors which differ by at least k
- adjacent vertices have colors which differ by at least h

the aim is to minimize λ

Calamoneri continuosly updates an annotated bibliografy
http://www.dsi.uniroma1.it/~calamo/survey.html

Variants of the problem

- minimize the order instead of the span, i.e. the effective number of used colors,
- is it possible to L(h,k)-labeling a graph using colors from 0 to a fixed given value?
- is it possible to find a no-hole L(h,k)-labeling ? i.e. a labeling using ALL the colors from 0 to λ
- is it possible to L(h,k)-labeling a graph extending the partial given coloring of some vertices?
- ....

Different approaches

graph theory and combinatorics

simulated annealing

genetic algorithms

neural networks

……………………

Special cases of h and k

L(1,0)-labeling	classical vertex coloring of G
L(0,1)-labeling	control code assignment
L(1,1)-labeling	classical vertex coloring of G^2
L(2,1)-labeling	radiocoloring problem

Is L(h,k) NP-complete?

L(h,k)	Conjectured in the general case (Fiala, Kloks, Kratochvil 2001) Proved on trees if k doesn't divide h (Fiala, Golovach, Kratochvil 2008)	
L(1,0)	(Karp 72)	Reduction from 3SAT
L(0,1)	(Bertossi, Bonuccelli 95)	Reduction from 3vertex coloring
L(1,1)	(McCormick 83)	Reduction from 3SAT
L(2,1)	(Griggs,Yeh 92)	Reduction from hamiltonian path

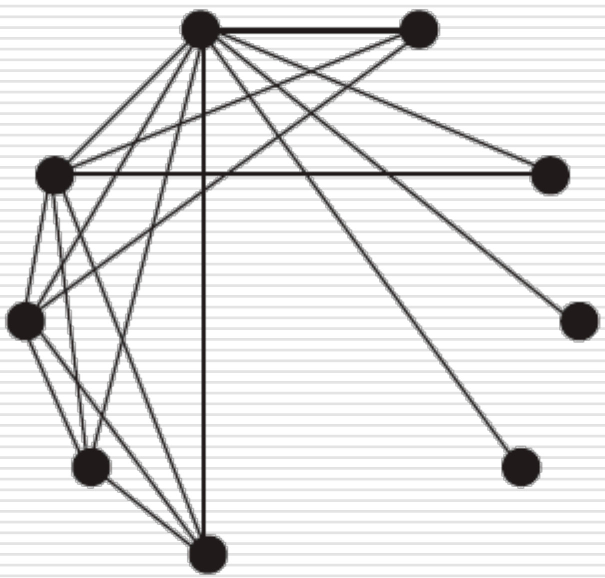
Open problem on $L(h,k)$

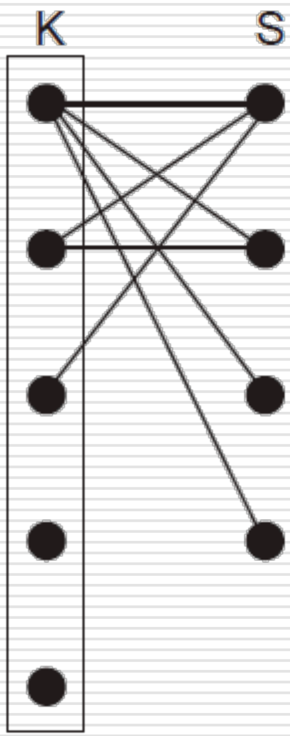
Is $L(h,k)$ NP-complete?

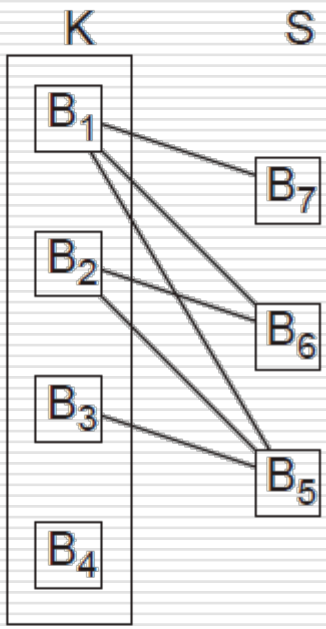
Threshold graphs

Threshold graph iff the vicinal preorder on V is total

(Chavatal, Hammer,Henderson, Zalcstein 1977)







N.V.R.Mahaved, U.N.Peled 1995 Threshold graphs and related topics

L(2,1)-labeling threshold graphs

In a threshold graph it always exists at least one universal vertex, hence it has diameter 2

diameter 2 graphs $\lambda_{2,1}(G) <= \Delta^2$(Griggs,Yeh 1992)

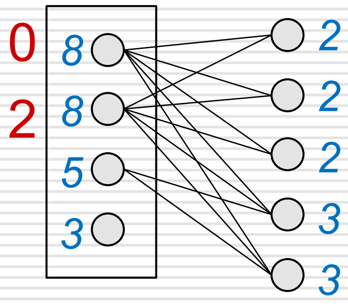
A threshold graph does not contain P_4 as subgraph, hence a TG is a subclass of cographs

linear time algorithm for cographs (Chang,Kuo 1996)

A new linear time algorithm for L(2,1)-labeling threshold graphs

Exact algorithm $\lambda_{2,1} \leq 2\Delta + 1$

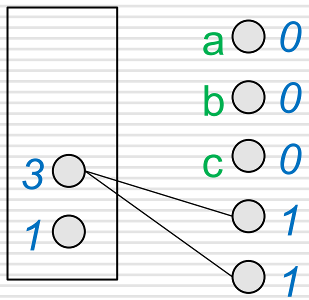
(Calamoneri,Petreschi 2006)



$8^2\ 5^1\ 3^3\ 2^3$

↑ Universal

D = {1,3}

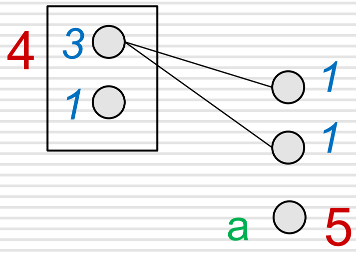


$3^1\ 1^3\ 0^3$

↑ Isolated

D = {1,3}

I = {a, b, c}

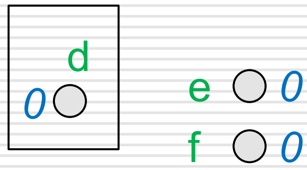


$3^1\ 1^3$

↑ Universal

D = {1,3}

I = {b, c}

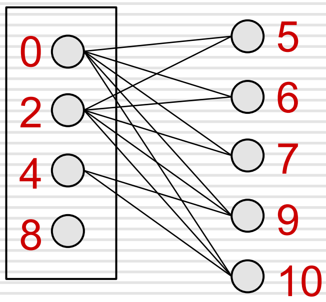


0^3

↑ Isolated

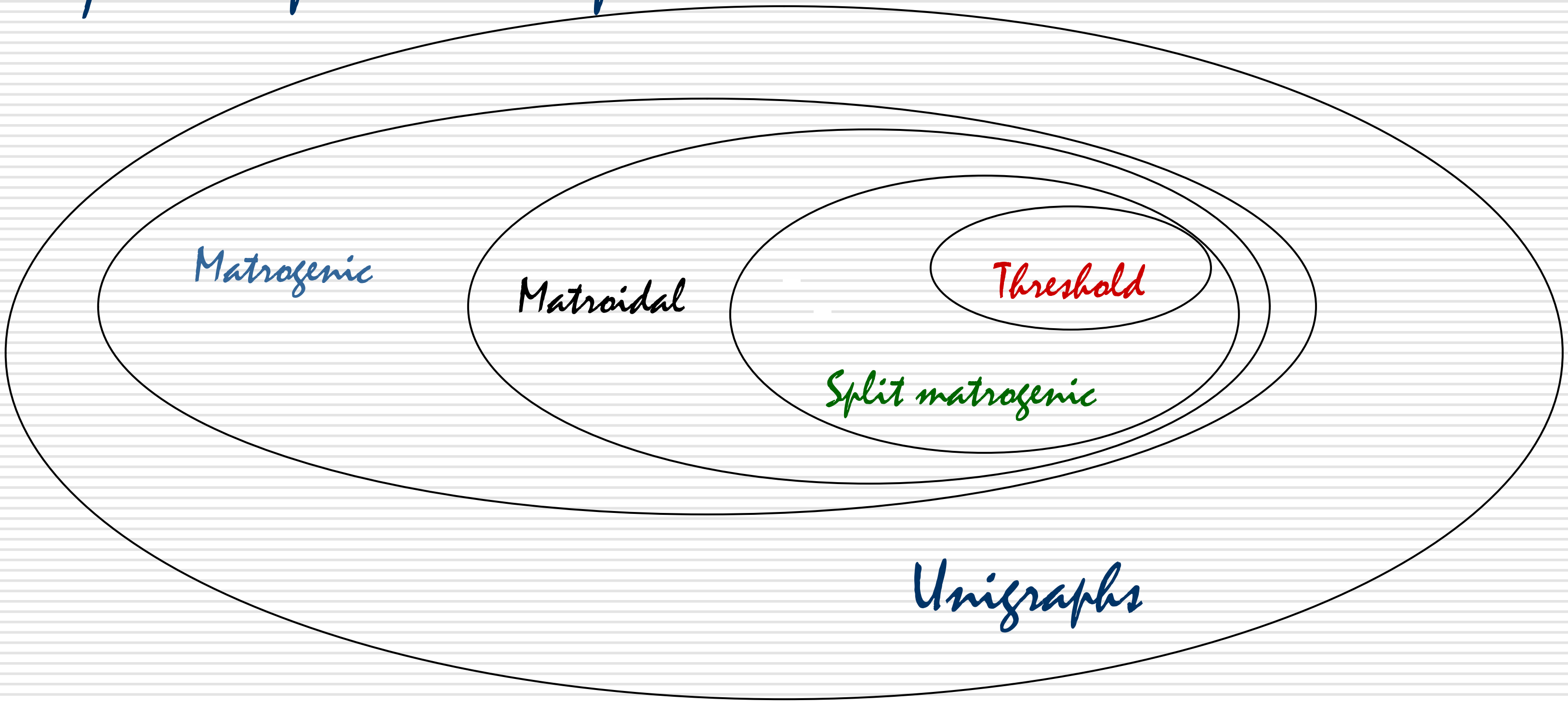
D = {1,3}

I = {b, c, d, e, f}



From threshold graphs to unigraphs

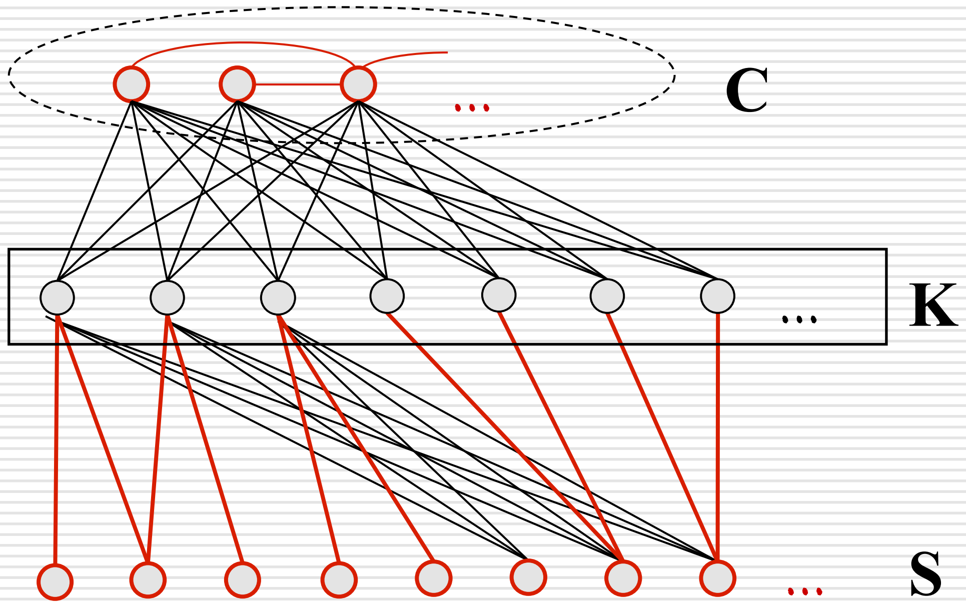
Unigraphs: graphs uniquely determined by their own degree sequence up to isomorphism



K,S,C

It is possible to define all the unigraphs as K U S U, by expanding the split-definition of threshold graph.

Borri, Calamoneri, Chavatal, Foldes, Hammer, Marchioro, Morgana, Orlin, Petreschi, Simeone, Tyshkevich (1977- 2009)



L(2,1)-labeling unigraphs

- Threshold

Exact algorithm $\lambda \leq 2\Delta + 1$

diameter 2 graphs $\lambda < \Delta^2$

- Split matrogenic

Approximate algorithm $\lambda \leq 3\Delta + 1$

split graphs $\lambda < \Delta^{1.5} + 2\Delta + 3$

- Matroidal

Approximate algorithm $\lambda \leq 3\Delta$

- Matrogenic

Approximate algorithm $\lambda \leq 3\Delta$

Exact algorithm $\lambda \leq 3\Delta$

- Unigraphs

Approximate algorithm

Boadlander et ot.. 2000/ Calamoneri, Petreschi 2006, 2010/ Cerioli, Posner 2010

Open problem on L(2,1)-labeling unigraphs

To prove (or disprove) the following conjecture:

The L(2,1)-labeling problem remains NP-hard for unigraphs

Conclusions

There is still a lot to say about L(2,1)-coloring of graphs!!!

Thanks for your attention!