Decidability of Interval Temporal Logics

Pietro Sala[1]

Departement Computer Science, University of Verona, Italy
pietro.sala@univr.it

ICTCS 2010

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Kamp and Reyle (1993)

"What is an interval?"

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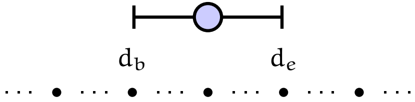

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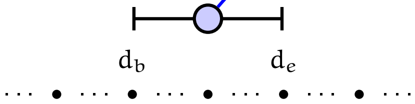

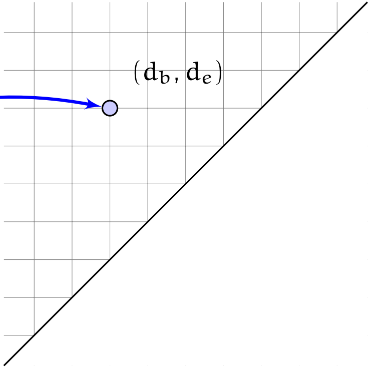

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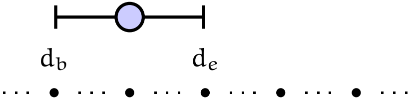

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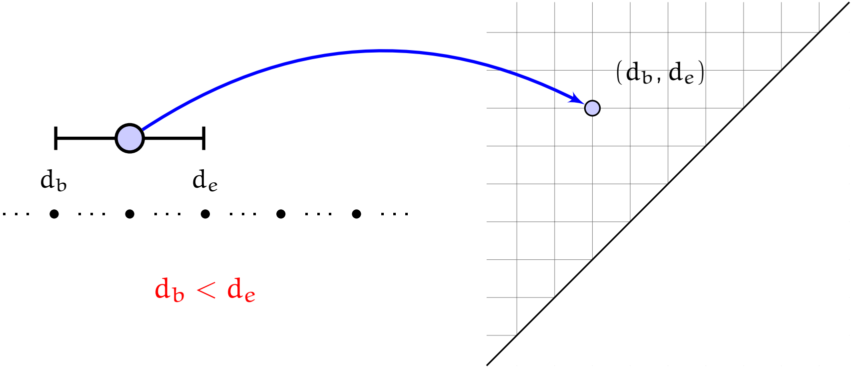

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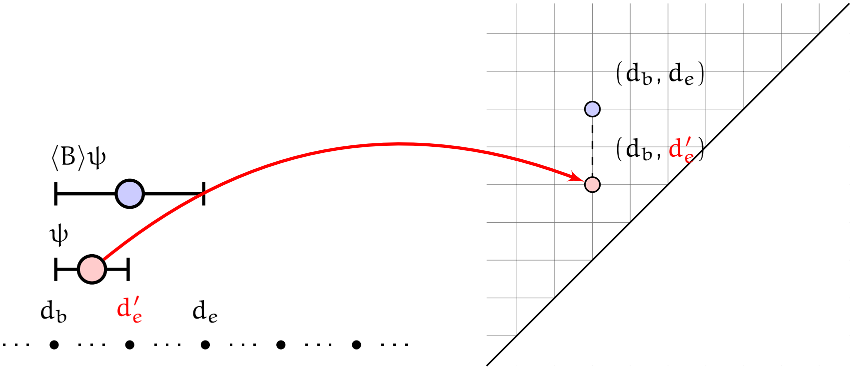

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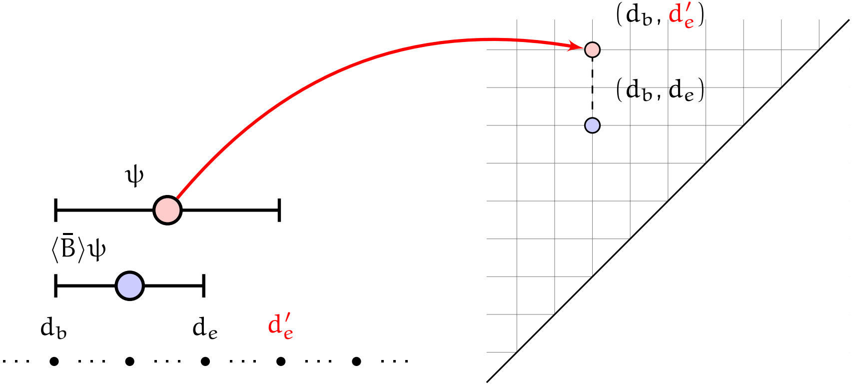

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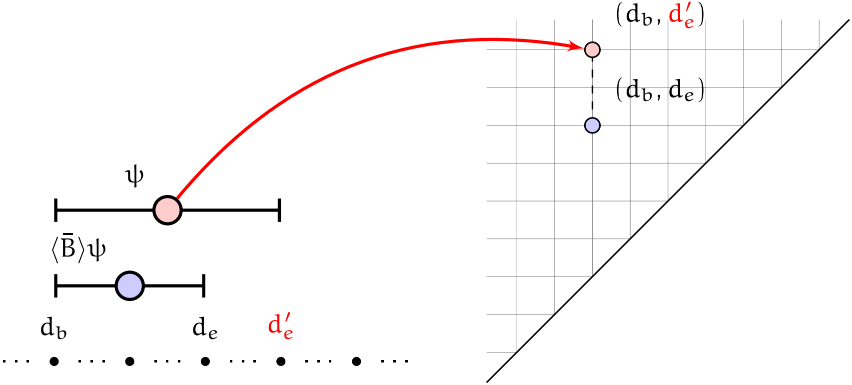

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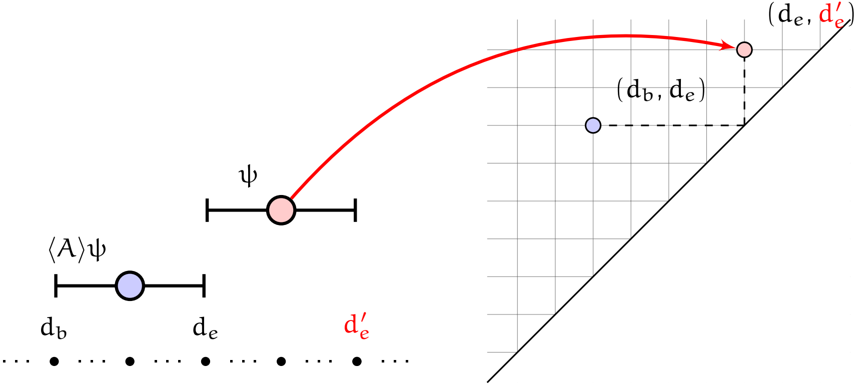

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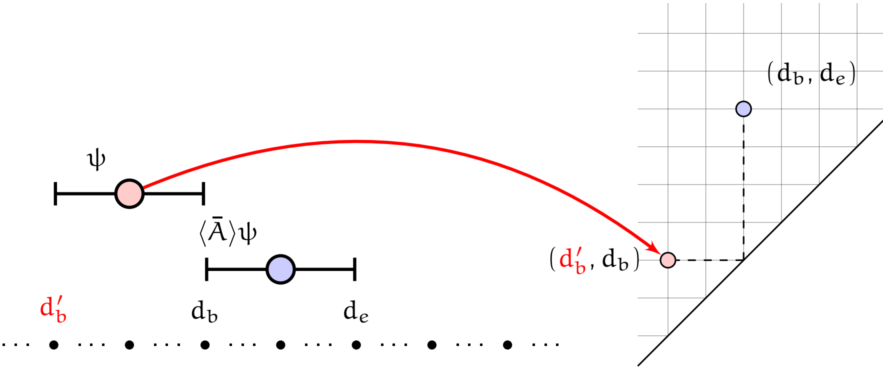

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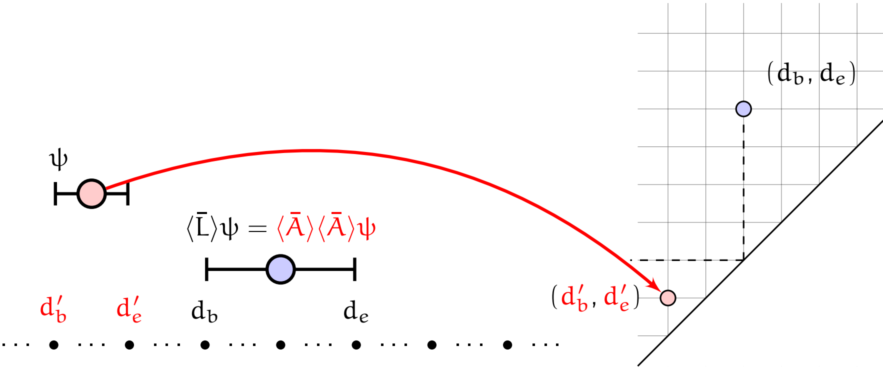

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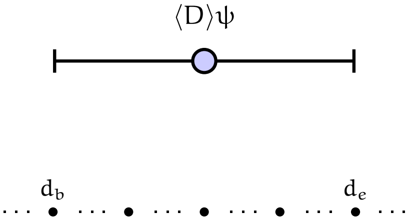

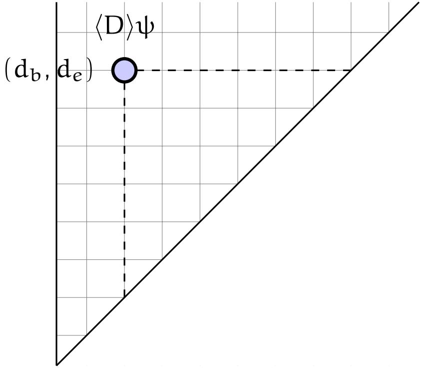

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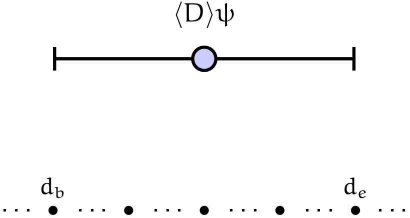

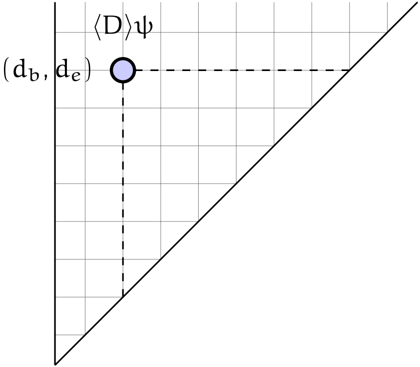

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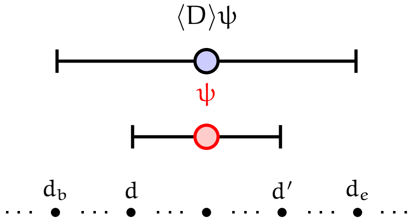

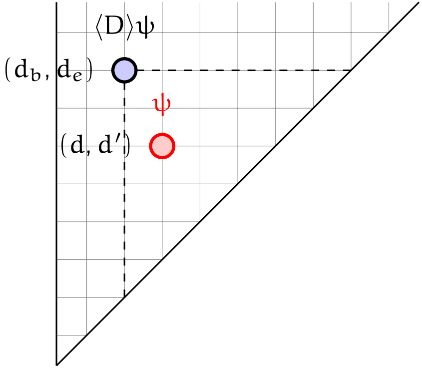

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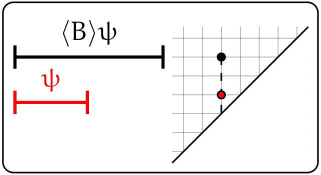
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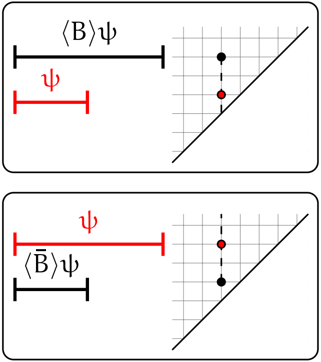

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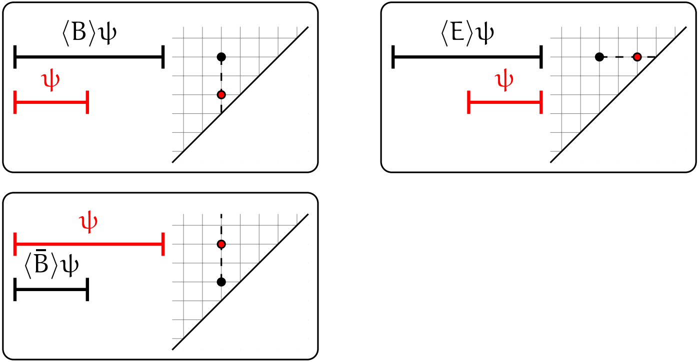
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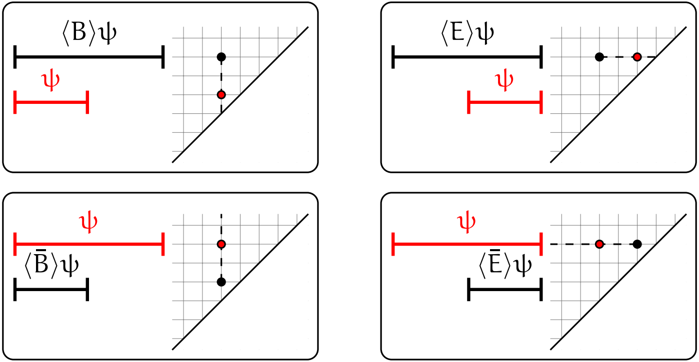

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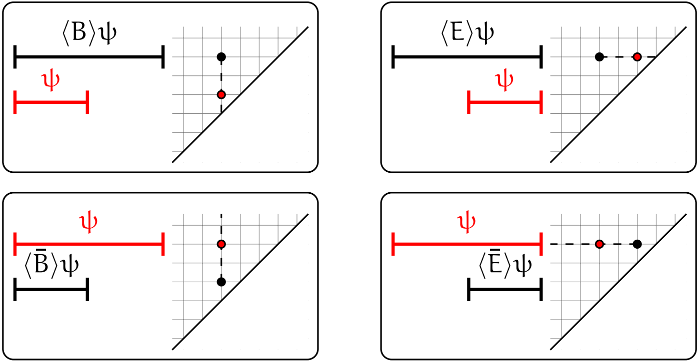
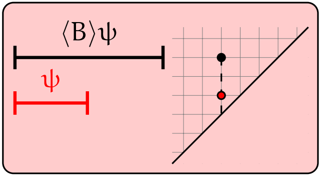


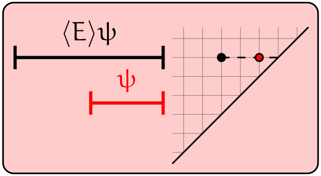
"The decision problem is solved when we know a procedure that allows, for any given logical expression, to decide by finitely many operations its validity or satisfiability. (...)The decision problem must be considered the main problem of mathematical logic."

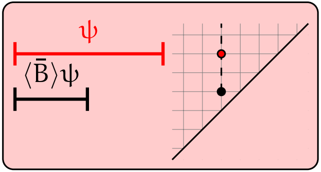
Hilbert and Ackermann (1928)

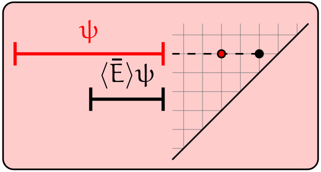
In principle, decidability of propositional interval logics depends on two factors:

- the set of interval modalities;
- the linear order over which the logic is interpreted.









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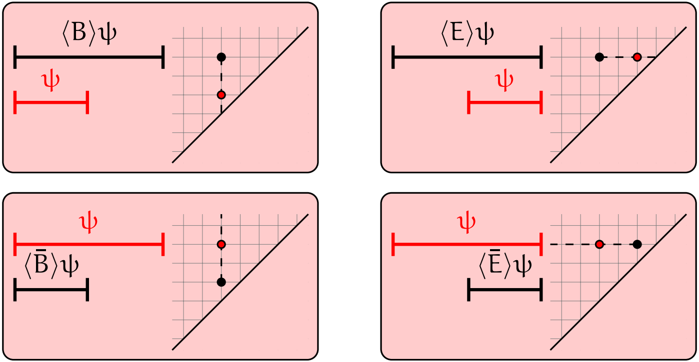

HS turns out to be undecidable over the class of linear orders under very weak assumptions.

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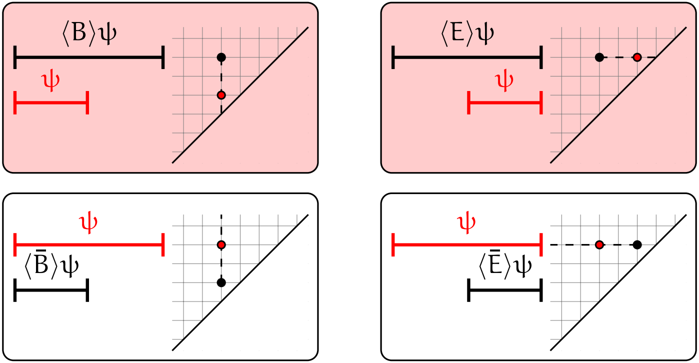

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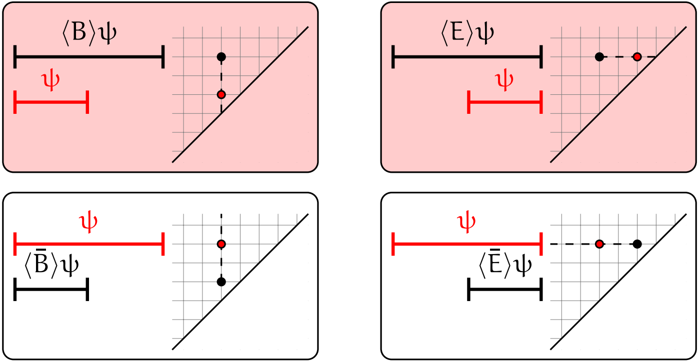

Lodaya proved that BE suffices to ensure undecidability (his proof is on the dense orders but can be easily adapted to the other class of linear orders).

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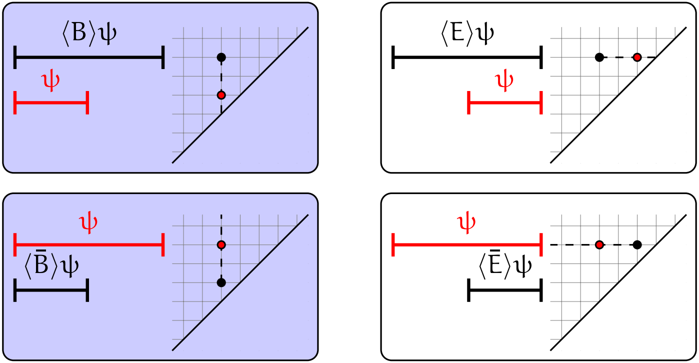

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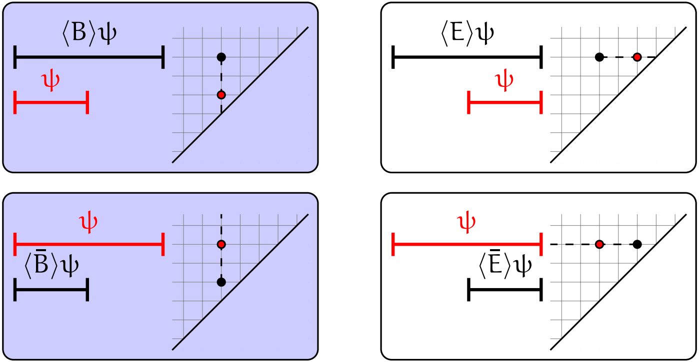

On the other hand the decidability of $B\bar{B}$ can be obtained in a straightforward way by reducing it to $\mathrm{LTL}[P, F]$.

Decidability

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These logics are maximal with respect to the decidability, in fact it has been proved that the addition of any other Allen's interval modality leads to an undecidable logic.

Moreover these logics cannot be expressed in a point based formalism.

How can we interpret a modal logic over the rational plane?

An example, Compass Logic:

$$\varphi \quad := \quad a \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi \quad | \quad \Diamond_{\rightarrow}\varphi \quad | \quad \Diamond_{\uparrow}\varphi \quad | \quad \Diamond_{\leftarrow}\varphi \quad | \quad \Diamond_{\downarrow}\varphi \quad |$$

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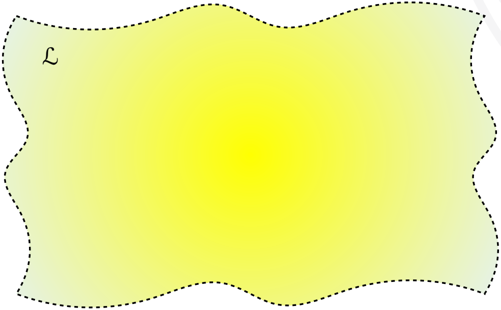

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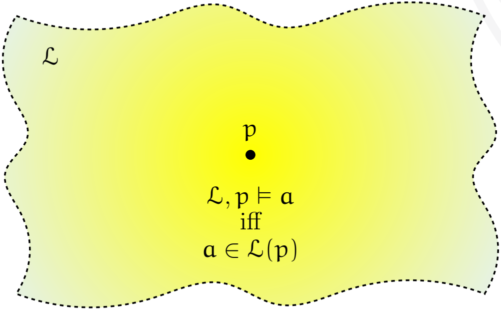

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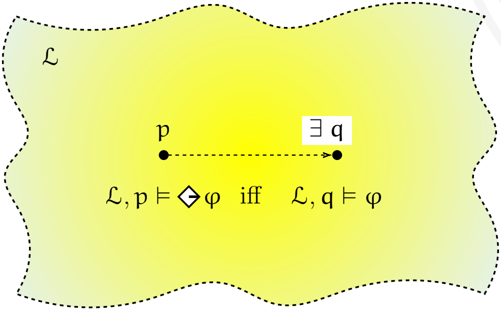

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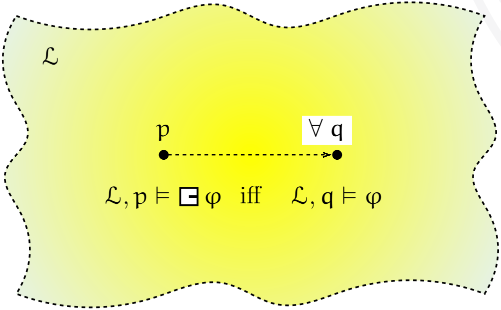

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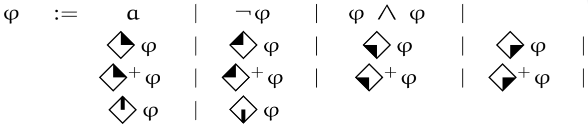

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$$
\begin{array}{lllllll}
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\end{array}
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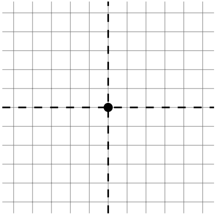

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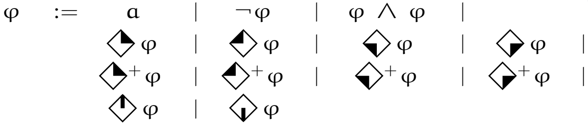

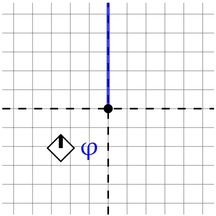

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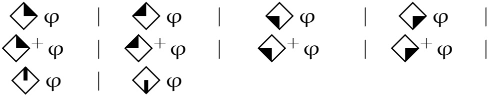

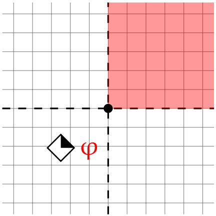

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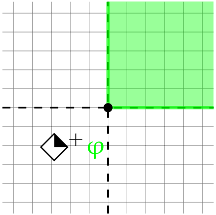
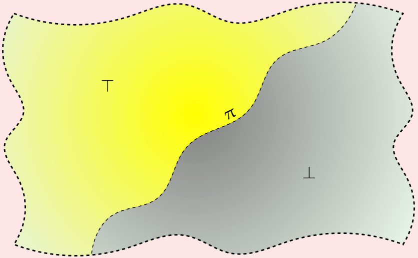


The "yellow region" of valid intervals (i.e., $\{p = (x, y) : x < y\}$) is (somehow) definable inside the dense plane:

$$\blacksquare(\top \vee \bot \vee \pi) \wedge \blacksquare(\neg\top \vee \neg\bot) \wedge \blacksquare(\neg\top \vee \neg\pi) \wedge \blacksquare(\neg\bot \vee \neg\pi) \wedge$$
$$\blacksquare(\top \rightarrow \Diamond_{\searrow}\top \wedge \text{◰}\,\top) \wedge \blacksquare(\bot \rightarrow \Diamond_{\nwarrow}\bot \wedge \text{◲}\,\bot) \wedge$$
$$\blacksquare(\pi \rightarrow \text{◰}^{+}\top \wedge \text{◲}^{+}\bot) \wedge \blacksquare(\Diamond_{\swarrow}\pi \wedge \Diamond_{\nearrow}\pi \rightarrow \pi \vee \Diamond_{\searrow}\pi \vee \Diamond_{\nwarrow}\pi).$$



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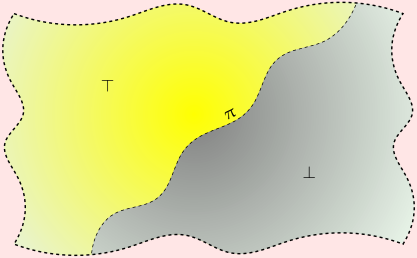
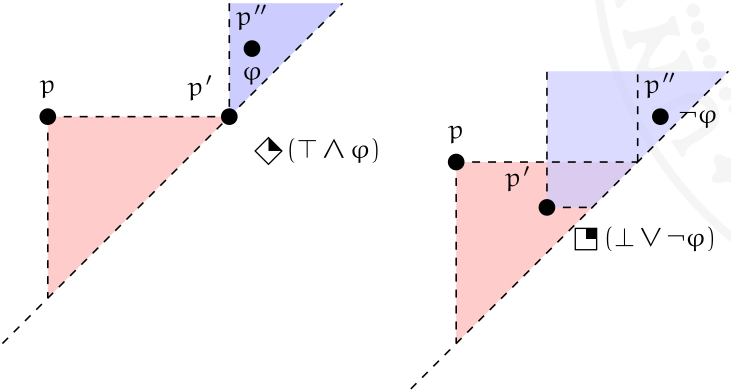
$$\blacksquare(\top \rightarrow \Diamond_{\searrow}\top \wedge \Box_{\nwarrow}\top) \wedge \blacksquare(\bot \rightarrow \Diamond_{\nwarrow}\bot \wedge \Box_{\searrow}\bot) \wedge$$
$$\blacksquare(\pi \rightarrow \Box_{\nwarrow}^{+}\top \wedge \Box_{\searrow}^{+}\bot) \wedge \blacksquare(\Diamond_{\swarrow}\pi \wedge \Diamond_{\nearrow}\pi \rightarrow \pi \vee \Diamond_{\searrow}\pi \vee \Diamond_{\nwarrow}\pi).$$



Then we can define $\langle D \rangle\varphi = \Diamond_{\searrow}(\top \wedge \varphi)$ and $\langle B \rangle\varphi = \Diamond_{\downarrow}(\top \wedge \varphi)$.

$$\langle L\rangle\varphi = \Box^{+}(\top \rightarrow \Diamond(\top \wedge \varphi)),\ [L]\varphi = \Diamond^{+}(\top \wedge \Box(\bot \vee \neg\varphi))$$



To solve the satisfiability problem for Cone Logic, we consider *portions* of the dense plane:

Stripe

A **stripe** of a labeled rational plane $\mathcal{L} : \mathbb{D} \times \mathbb{D} \rightarrow A$ is the restriction $\mathcal{L}_{[x_0,x_1]}$ of $\mathcal{L}$ to a region of the form $[x_0, x_1] \times \mathbb{D}$.

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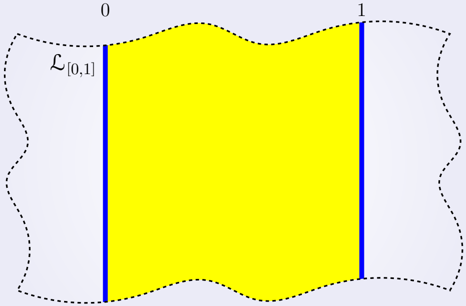
Fact

Any Cone Logic formula φ can be translated into a formula $\varphi_{[x_0, x_1]}$ in such a way that, for every labeled dense plane $\mathcal{L}$,

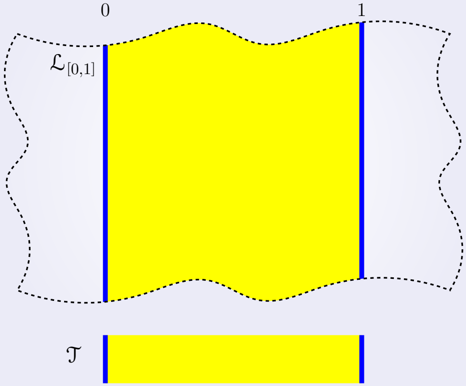
$$\begin{array}{c} \exists\, p \in \mathbb{D} \times \mathbb{D}. \\ \mathcal{L}, p \models \varphi \end{array} \quad \text{iff} \quad \begin{array}{c} \exists\, p \in [x_0, x_1] \times \mathbb{D}. \\ \mathcal{L}_{[x_0, x_1]}, p \models \varphi_{[x_0, x_1]}. \end{array}$$

Decompositions of stripes



By exploiting isomorphism between the orders
over $[0,1]$ and over $\left\{\frac{i}{2^n} : n \in \mathbb{N},\ 0 \leqslant i \leqslant 2^n\right\}$,
we decompose the stripe $\mathcal{L}_{[0,1]}$ into a **tree structure** $\mathcal{T}$...

Decompositions of stripes



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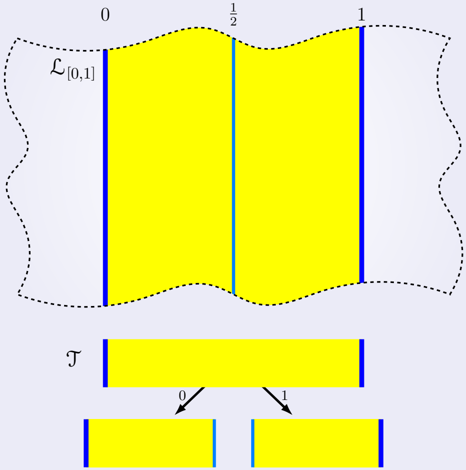

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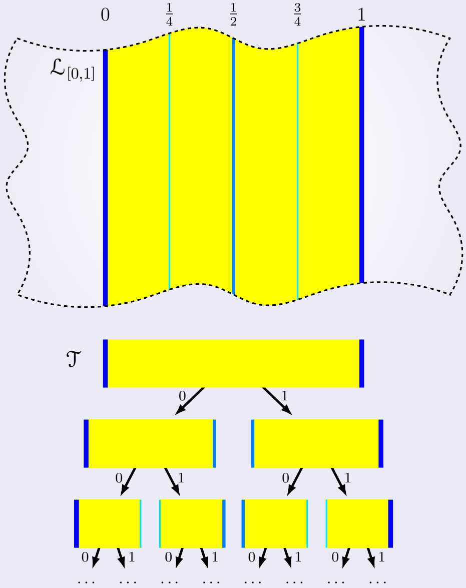

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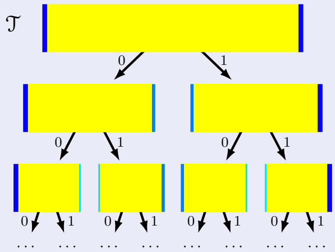

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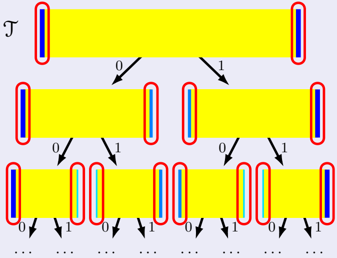

Finite abstractions of stripes

Given a (sub-)stripe $\mathcal{L}_{[x_0,x_1]}$, we define an equivalence $\approx$ over $\mathbb{D}$ such that $y \approx y'$ iff, for every *subformula* α of φ, we have

$$\mathcal{L},(x_0,y) \models \Diamond\alpha \;/\; \square\alpha \;/\; \Diamond\alpha \;/\; \square\alpha \qquad \mathcal{L},(x_1,y) \models \Diamond\alpha \;/\; \square\alpha \;/\; \Diamond\alpha \;/\; \square\alpha$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

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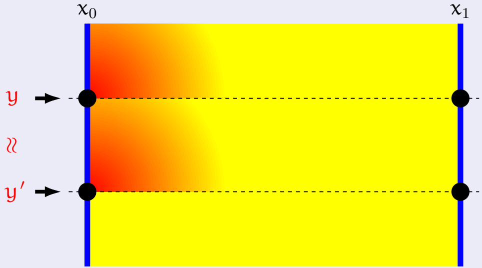

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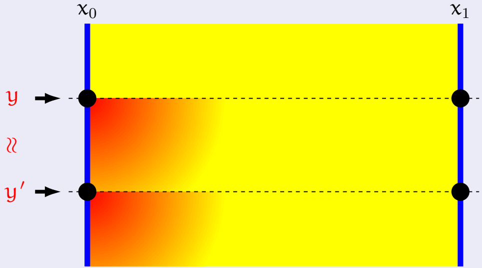

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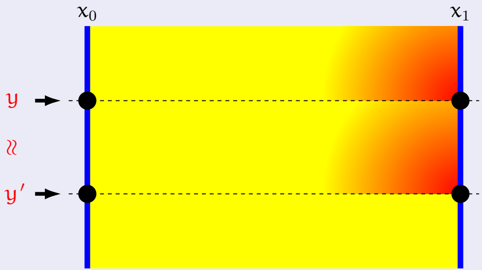

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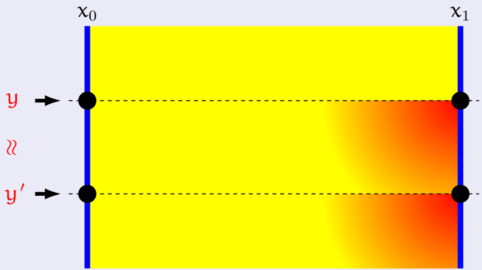

Since the equivalence relation $\approx$ has *finite index*, we have that

Proposition (a tree pseudo-model property)

Any given stripe $\mathcal{L}_{[0,1]}$ can be represented by means of a suitable *infinite binary tree* $\mathcal{T}$ whose vertices are labeled over a finite alphabet (we call the structure $\mathcal{T}$ a **tree decomposition** of $\mathcal{L}_{[0,1]}$).

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...However, tree decompositions must be properly constrained so that they correctly represents some concrete stripes.

Examples of constraints on a tree decomposition

- For every pair of *sibling* vertices $v = [x_0, x_1]$ and $v' = [x'_0, x'_1]$ in $\mathcal{T}$, the labeling of the right border of v has to *match* with the labeling of the left border of v' (in such a way, we can assume $x_1 = x'_0$),
- There is *no infinite path* π in $\mathcal{T}$ such that, for every vertex $v \in \pi$, $\Diamond\alpha$ appears on the *left border* of v and neither $\Diamond\alpha$ nor α appear on the *right border* of v.

Theorem 1 (reduction to a CTL fragment)

Constrained tree decompositions can be defined in a **fragment of CTL**, which we denote $\mathbf{CTL}^{-}$.

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Constrained tree decompositions can be defined in a **fragment of CTL**, which we denote **CTL$^-$**.

Theorem 2 (deciding satisfiability of CTL$^-$)

The satisfiability problem for CTL$^-$ is decidable in **PSPACE**.

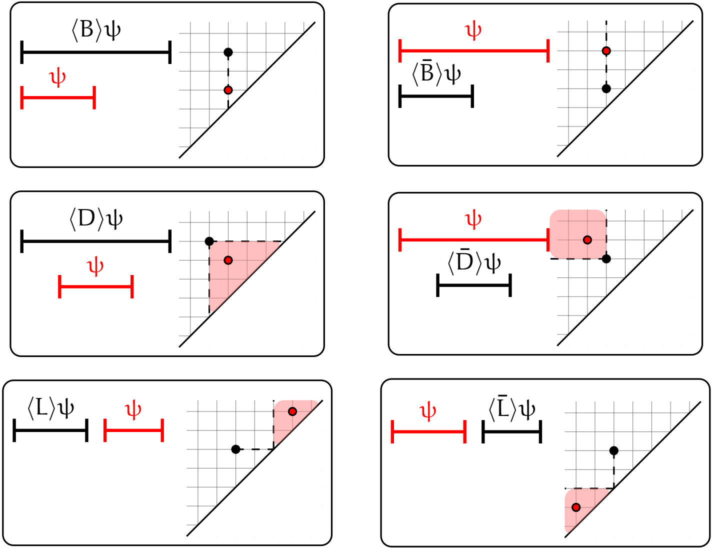
Proof idea

CTL$^-$ formulas are *conjunctions* of the following basic formulas:

1. $\mathbf{AG}(\mathit{left} \vee \mathit{right})$, $\mathbf{AG}\neg(\mathit{left} \wedge \mathit{right})$, $\mathbf{AG}(\mathbf{EX}\mathit{left} \wedge \mathbf{EX}\mathit{right})$
2. $\mathbf{AG}(\xi)$, $\mathbf{AG}(\rho \rightarrow \mathbf{AF}\xi)$,
   where ρ is a simple propositional formula and ξ contain only *positive* occurrences of $\mathbf{AX}$ and no occurrences of other operators.

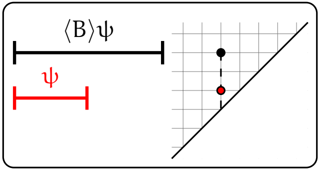
$\Rightarrow$ Checking satisfiability of these formulas amounts at deciding **universality of Büchi automata**.

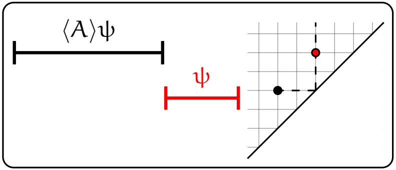
We show that the decidability of Cone Logic subsumes the decidability of the following logic over dense orderings:

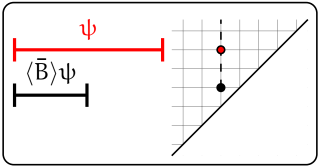


Another (maximal) interval temporal logic

Consider now the logic $AB\bar{B}\bar{A}$ interpreted over FINITE linear orders.







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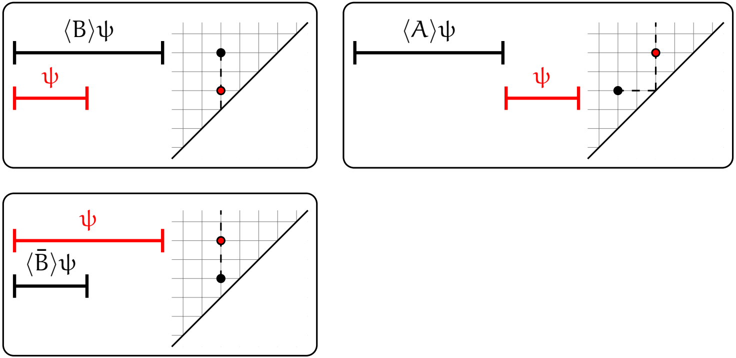

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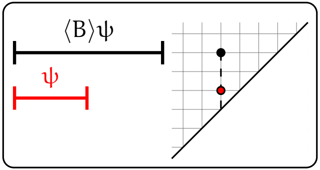

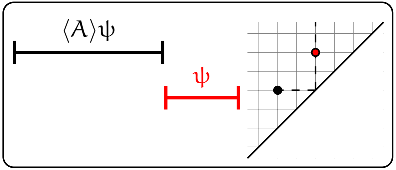

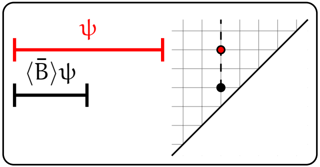

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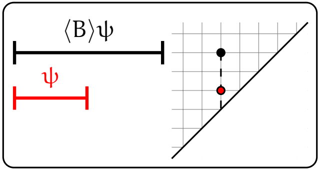

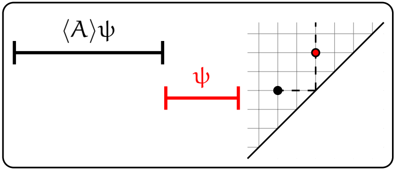

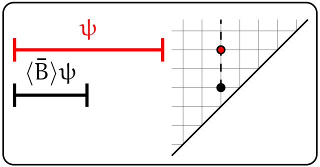

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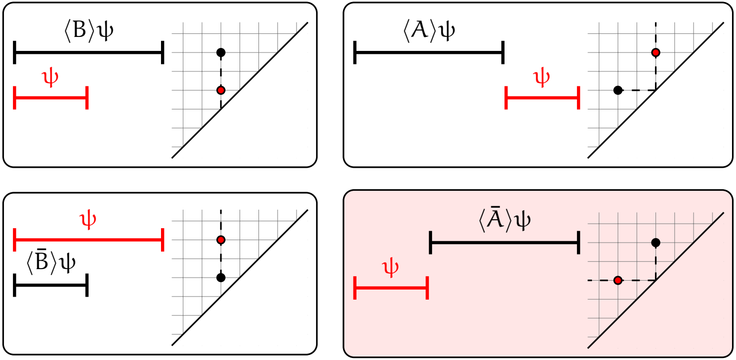
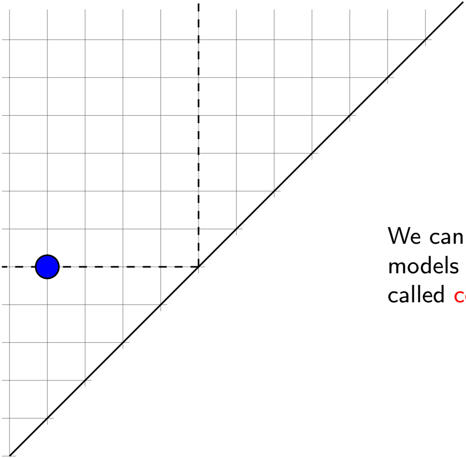


A spatial account of models



We can give a spatial interpretation to models of a formula φ as colored octants, called compass structures:

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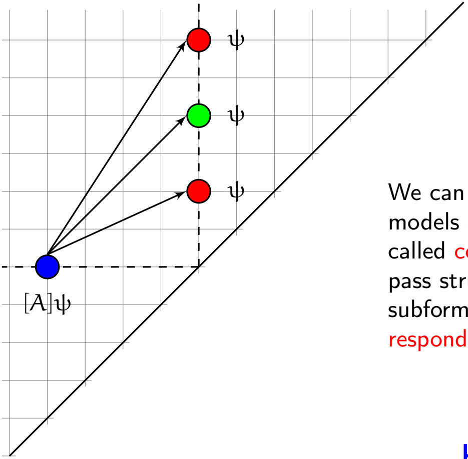
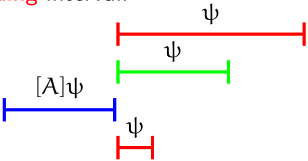
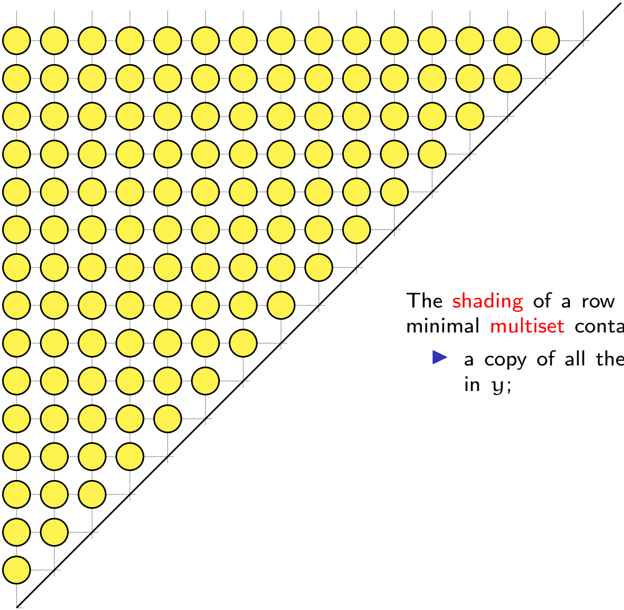


We can give a spatial interpretation to models of a formula φ as colored octants, called compass structures: points of compass structures are colored with the set of subformulas of φ that are true over the corresponding interval.



Decidability over finite linear orders



The shading of a row $\mathfrak{y}$ ($\mathcal{S}\mathfrak{h}(\mathfrak{y})$) is a minimal multiset containing:

- a copy of all the colors which occur in $\mathfrak{y}$;

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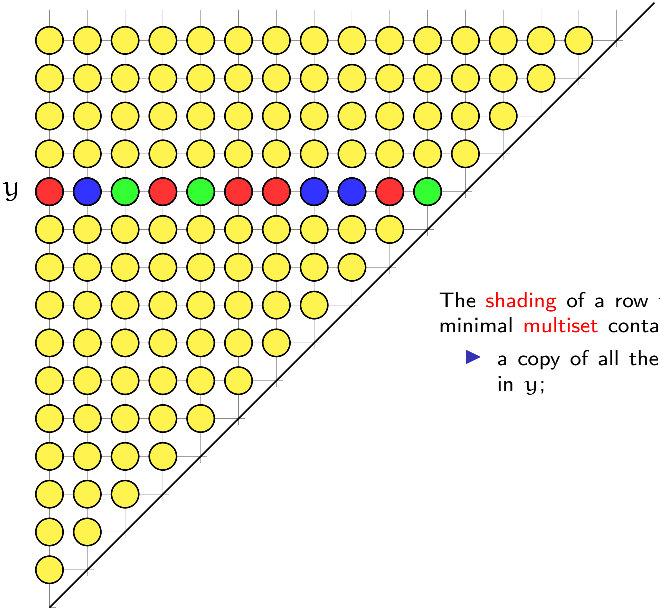

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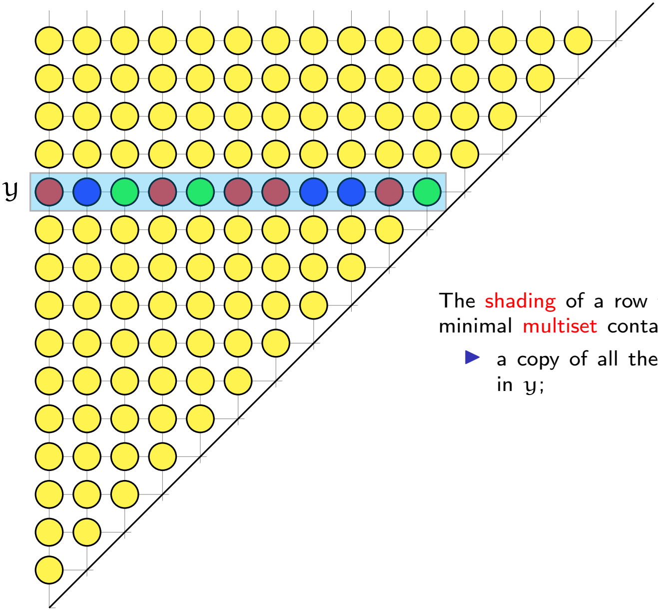

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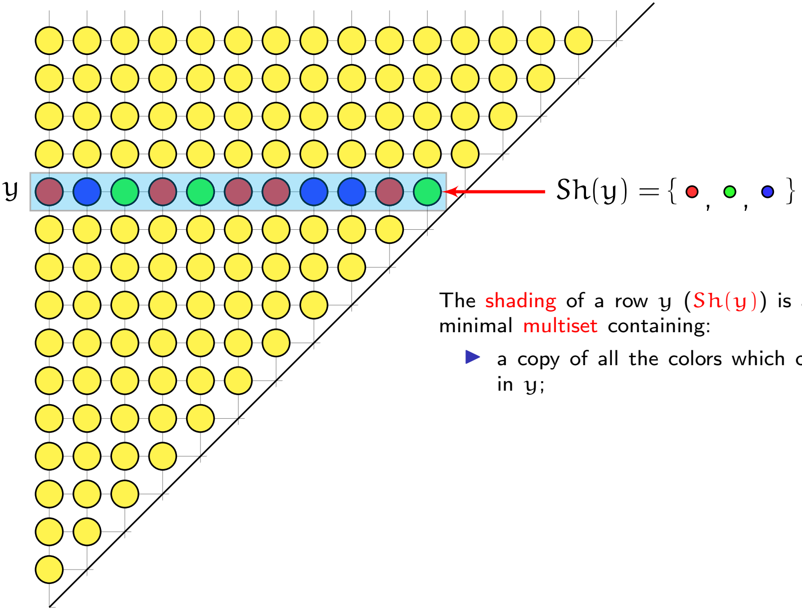

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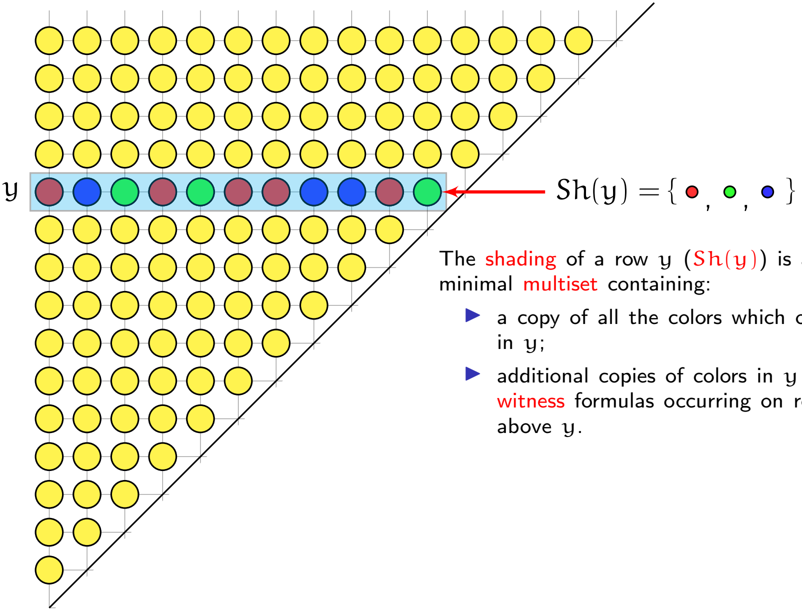

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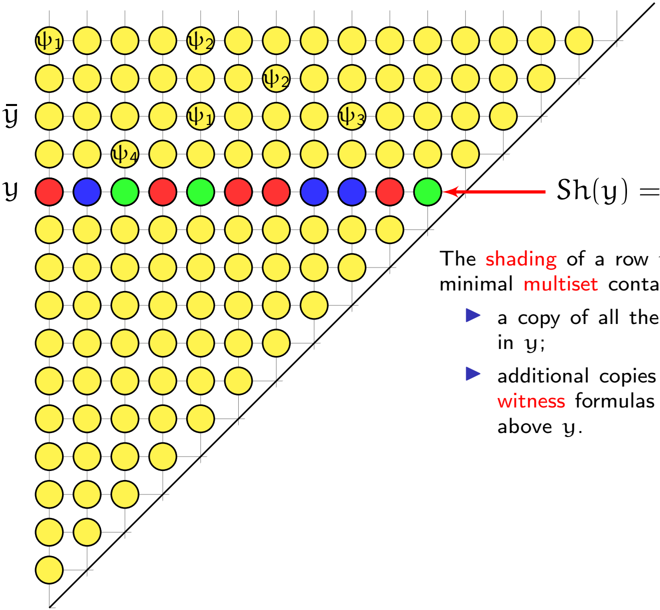

$\mathcal{S}h(y) = \{ \bullet, \bullet, \bullet \}$

The shading of a row y ($\mathcal{S}h(y)$) is a minimal multiset containing:

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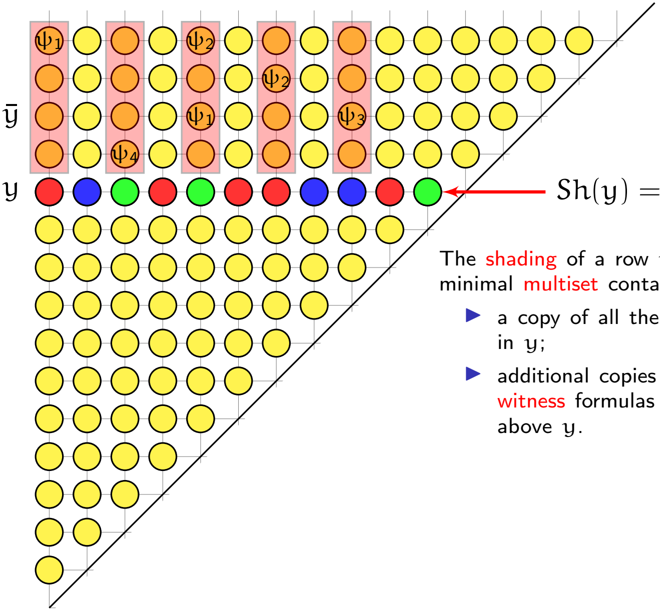

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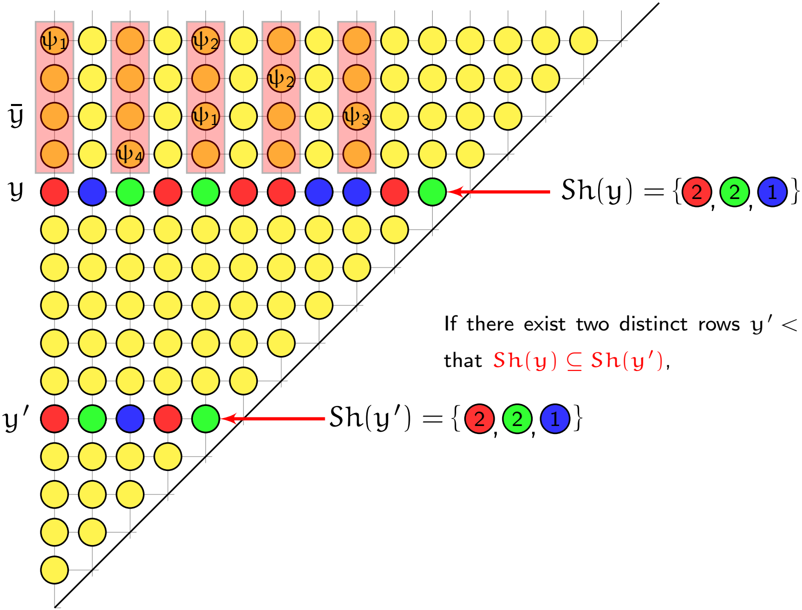

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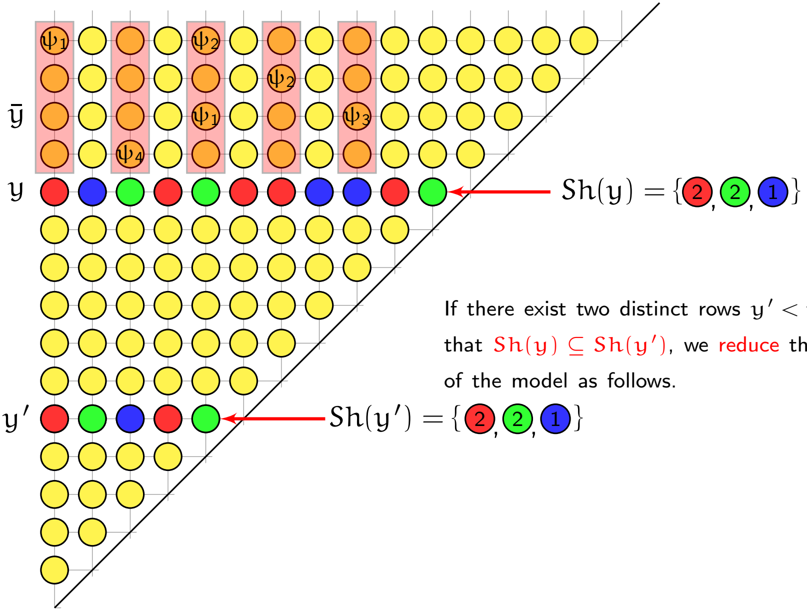

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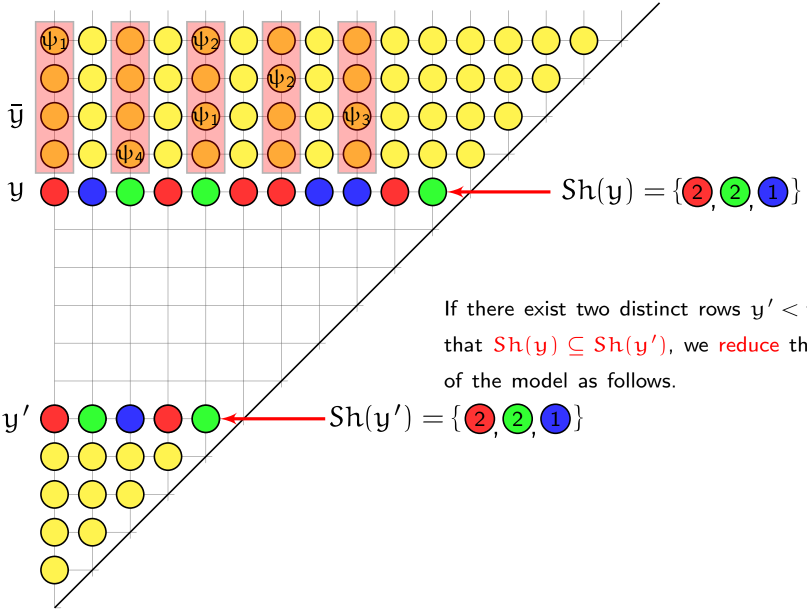

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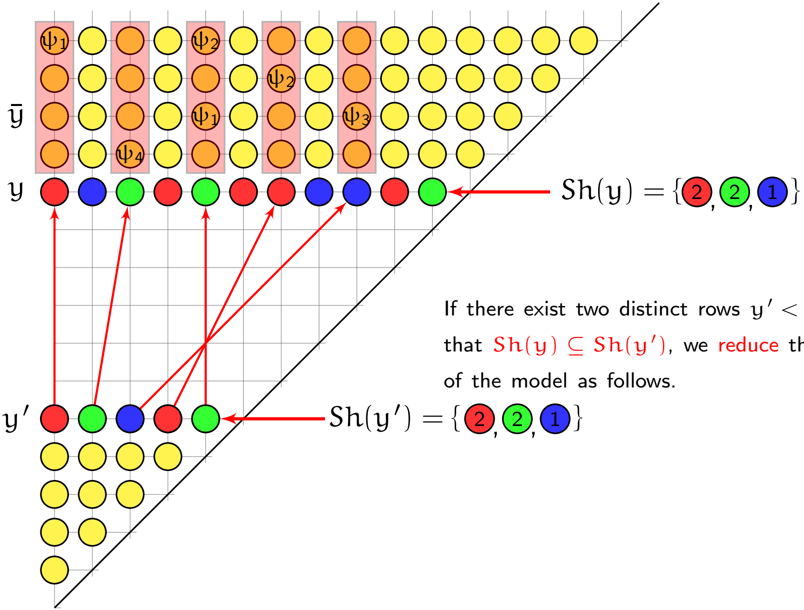

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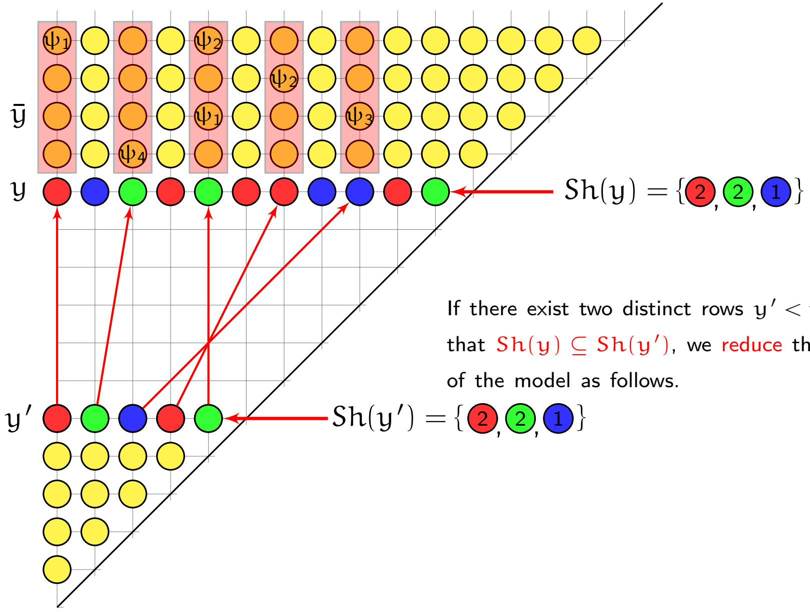

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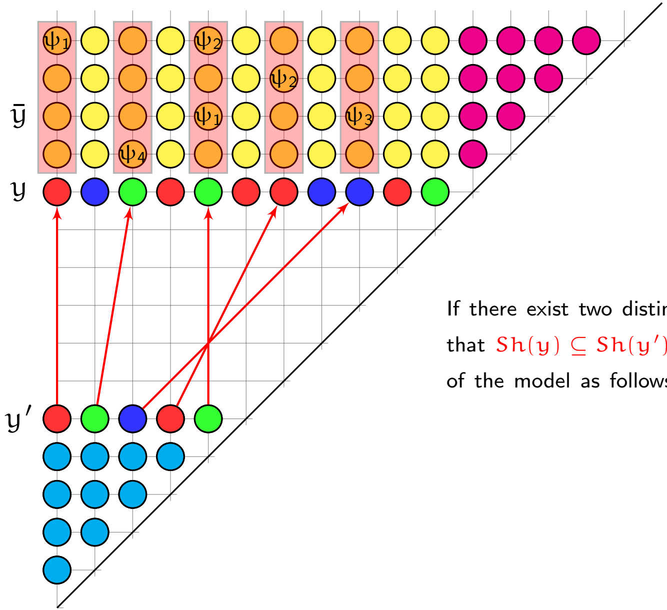

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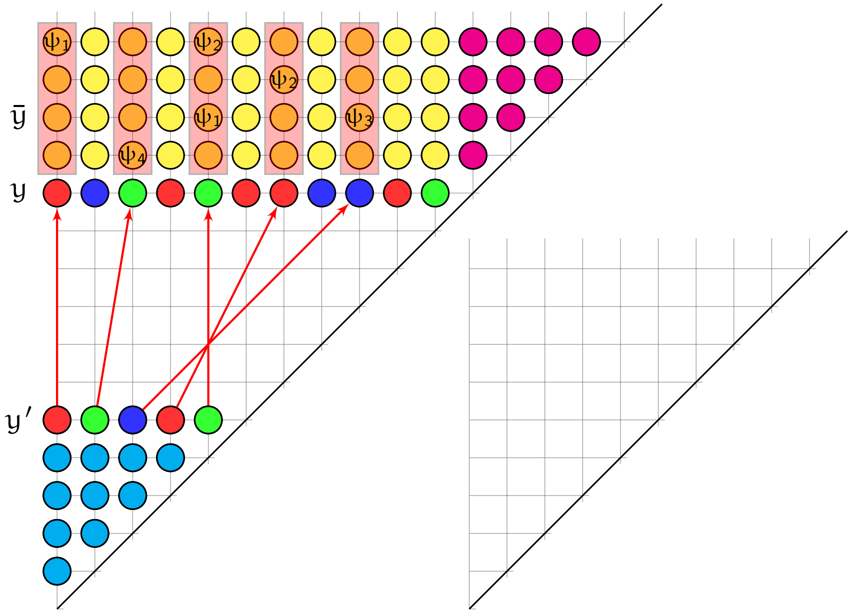

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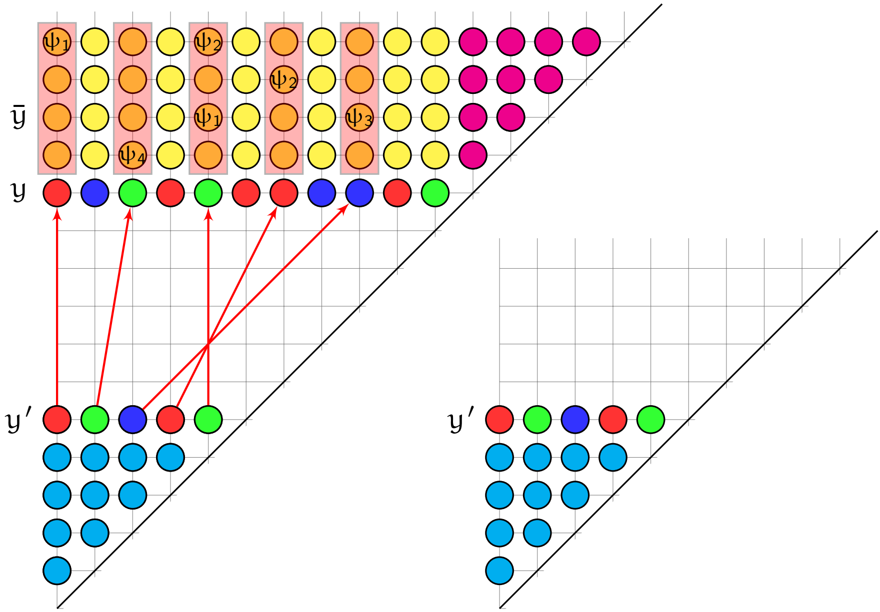

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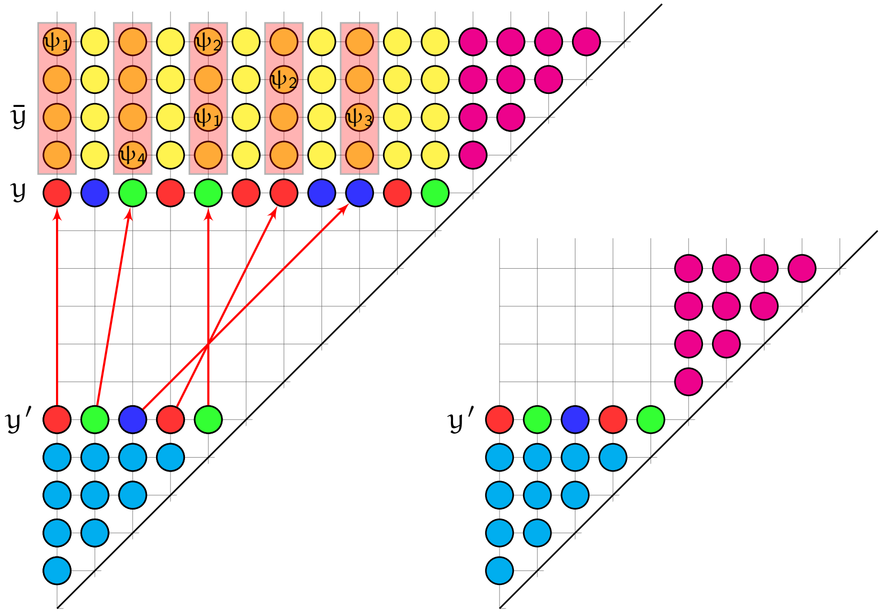

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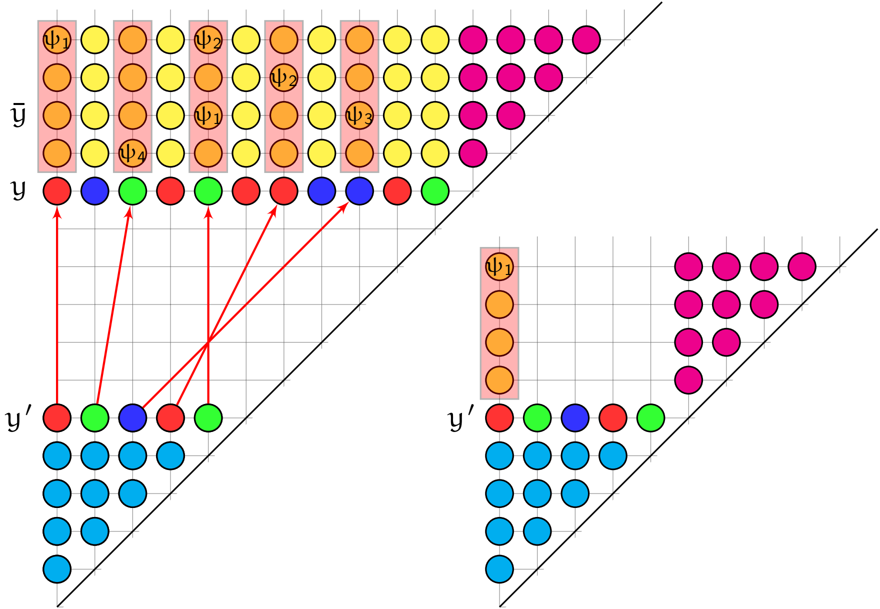

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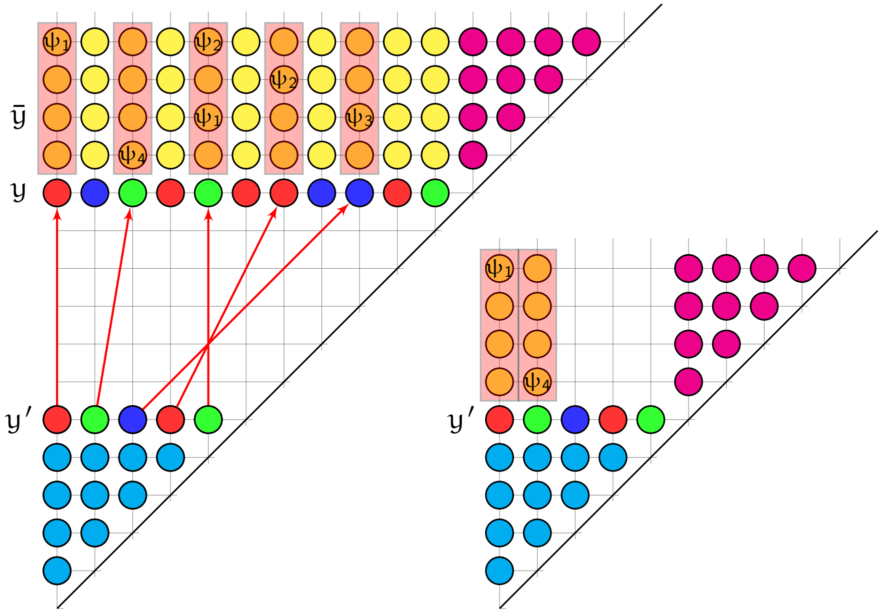

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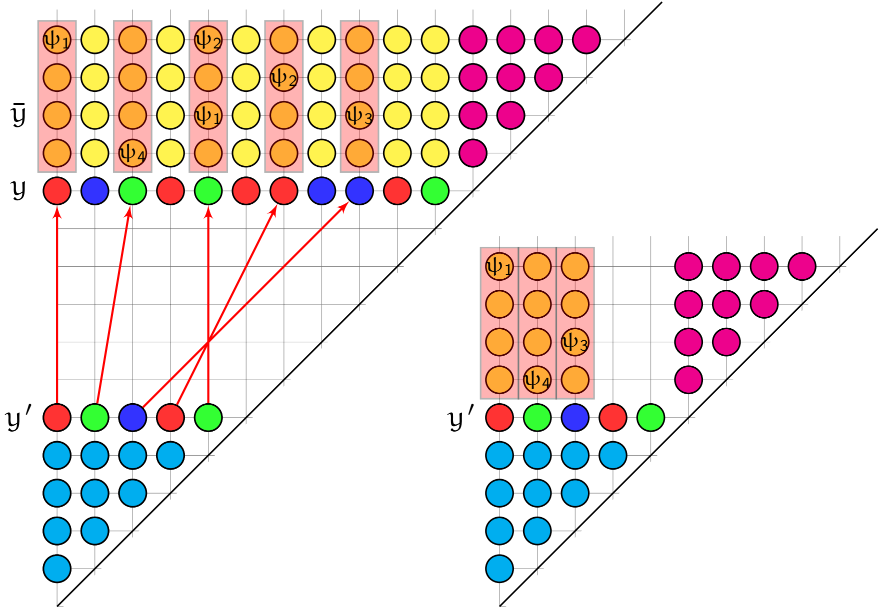

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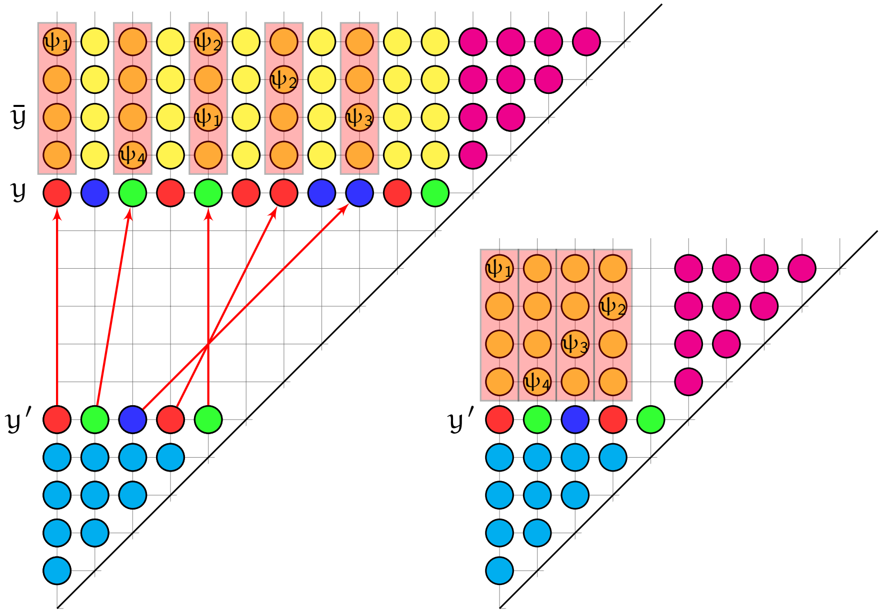

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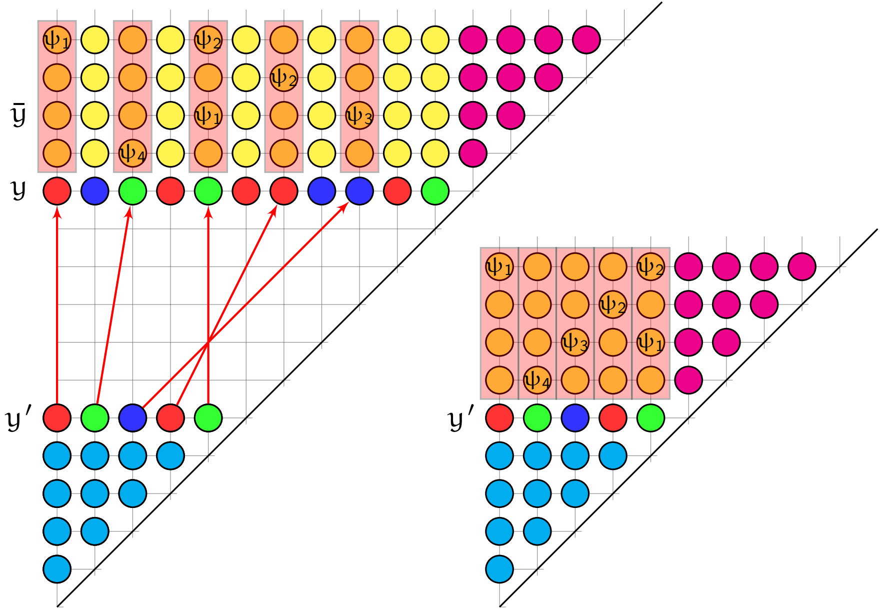
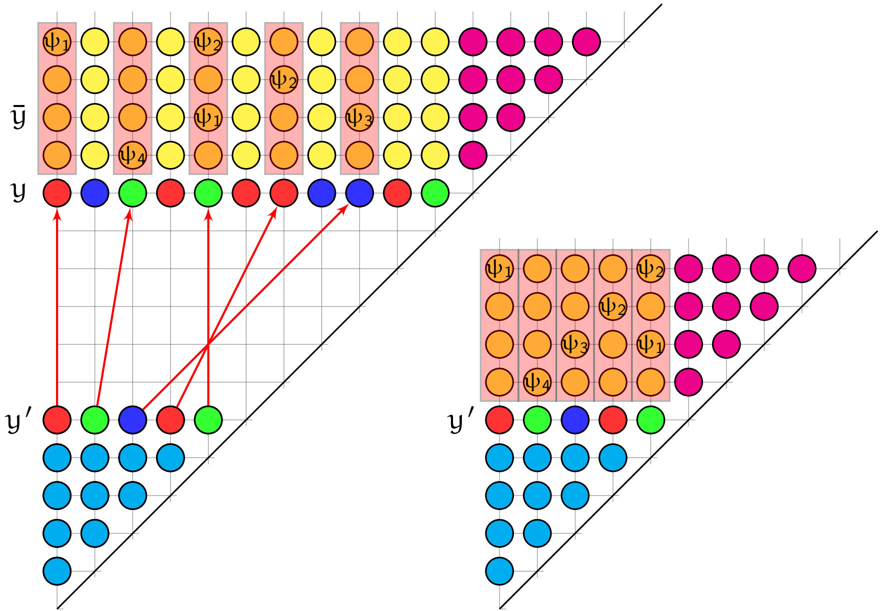


Decidability over finite linear orders



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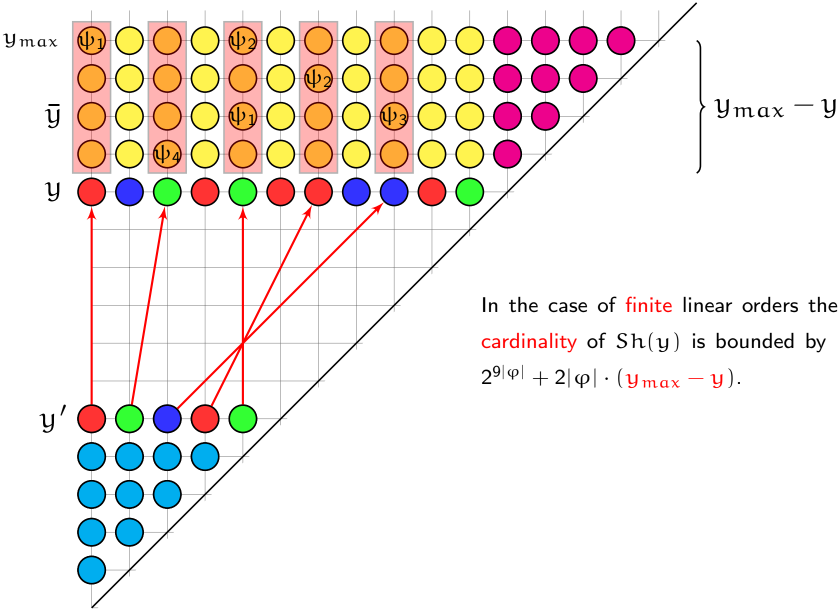
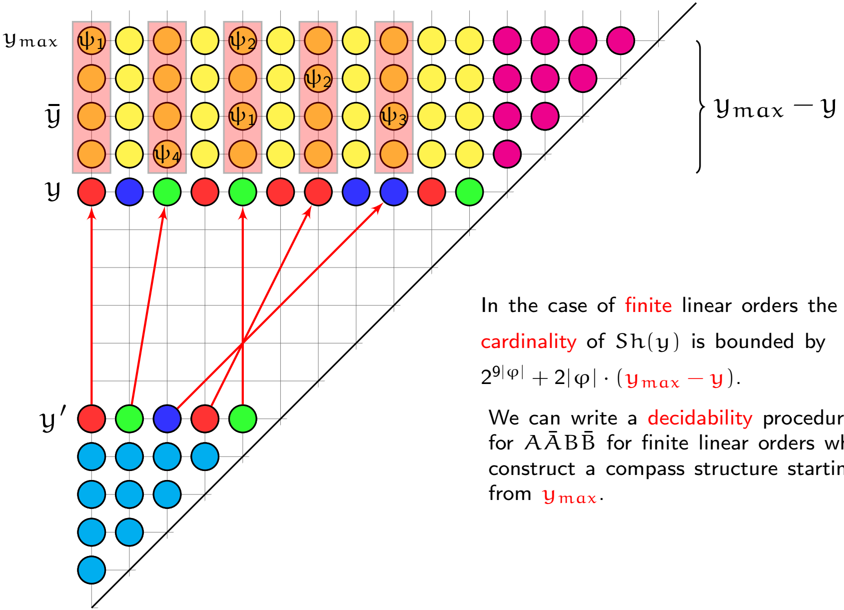


In the case of finite linear orders the cardinality of $Sh(y)$ is bounded by $2^{9|\varphi|} + 2|\varphi| \cdot (y_{max} - y)$.

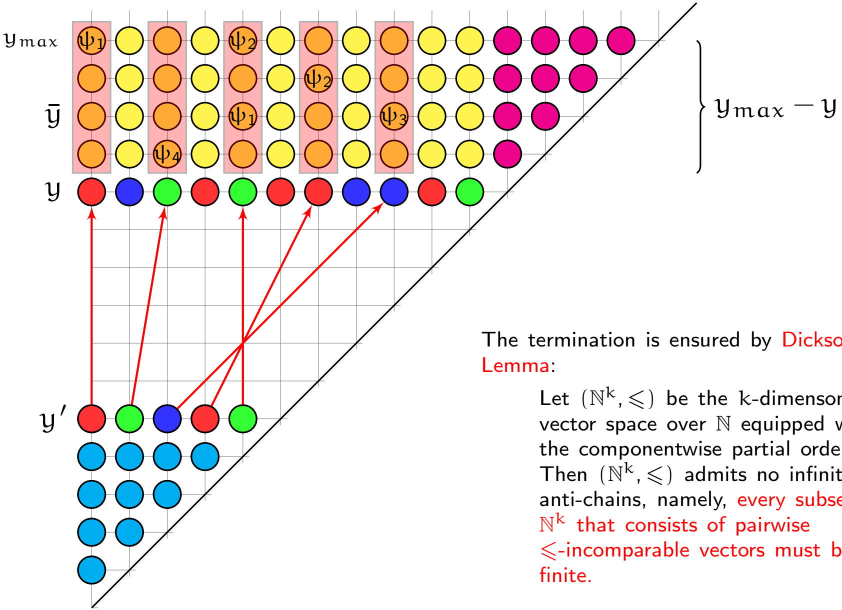
Decidability over finite linear orders



In the case of finite linear orders the cardinality of $Sh(y)$ is bounded by $2^{9|\varphi|} + 2|\varphi| \cdot (y_{max} - y)$.

We can write a decidability procedure for $A\bar{A}B\bar{B}$ for finite linear orders which construct a compass structure starting from y_{max}.

Decidability over finite linear orders

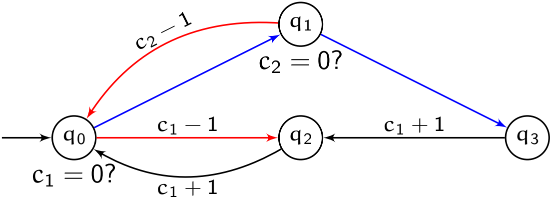


The termination is ensured by Dickson's Lemma:

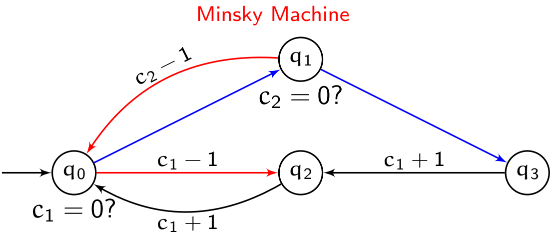
> Let $(\mathbb{N}^k, \leqslant)$ be the k-dimensonal vector space over $\mathbb{N}$ equipped with the componentwise partial order $\leqslant$. Then $(\mathbb{N}^k, \leqslant)$ admits no infinite anti-chains, namely, every subset of $\mathbb{N}^k$ that consists of pairwise $\leqslant$-incomparable vectors must be finite.

Complexity, Decidability and Undecidability

Minsky Machine

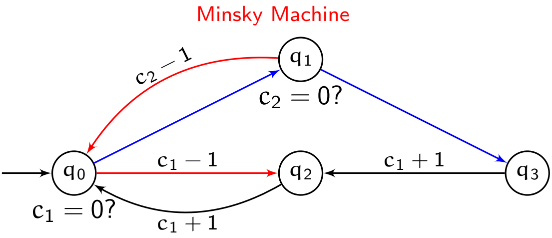


Complexity, Decidability and Undecidability



First we encode the configuration of a Minsky Machine by forcing every unit length interval to satisfy exact one among the letters $c_1, ..., c_n, q_0, ..., q_m$

Complexity, Decidability and Undecidability



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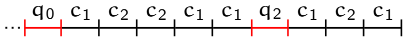
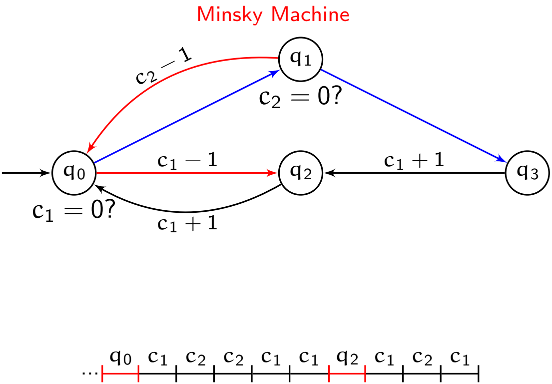
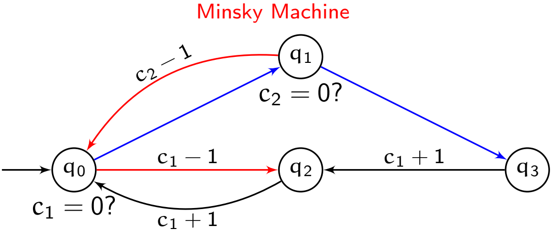


Complexity, Decidability and Undecidability

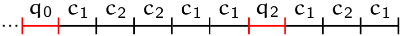


The value of the counter c_i is the number of the c_i-labeled unit intervals between two consecutive state-labeled unit intervals.

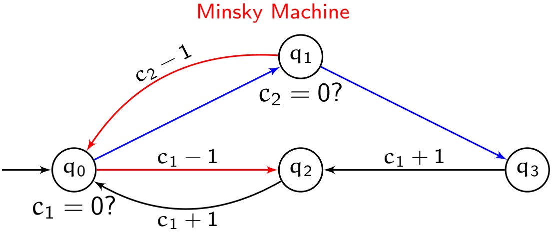
Complexity, Decidability and Undecidability

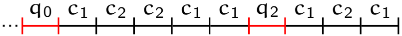


The crucial part of the encoding is maintaining the correct value of the counters between two consecutive configurations.



Complexity, Decidability and Undecidability

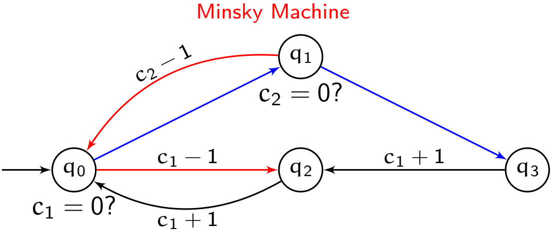


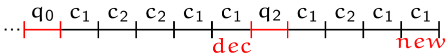


The crucial part of the encoding is maintaining the correct value of the counters between two consecutive configurations.

We can use a propositional letter p which behave like a function between the c_i-labeled intervals in two consecutive configurations.

Complexity, Decidability and Undecidability





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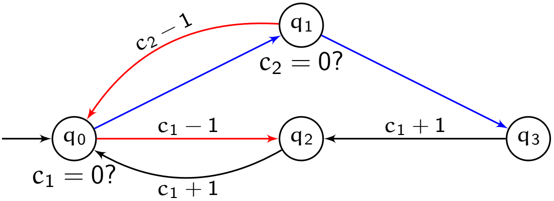
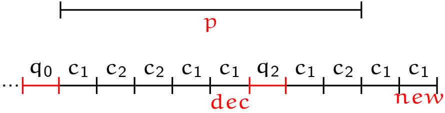
Complexity, Decidability and Undecidability

Minsky Machine



$$p \rightarrow \bigvee_i (\langle B\rangle c_i \wedge \langle A\rangle (c_i \wedge \neg new)) \wedge$$

$$\langle B\rangle\langle A\rangle \bigvee_i (q_i \wedge [B] \bigvee_i c_i) \wedge [B](\neg p \wedge \neg dec) \wedge [\bar{B}]\neg p$$



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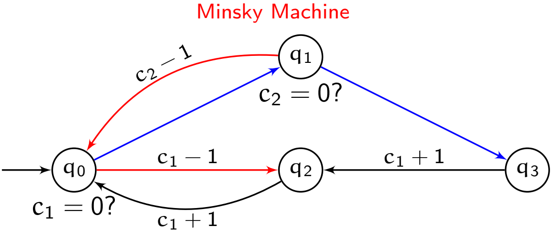

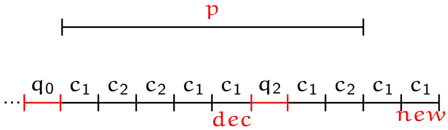

$\bigvee_i c_i \wedge \neg new \to \langle \bar{A} \rangle p$

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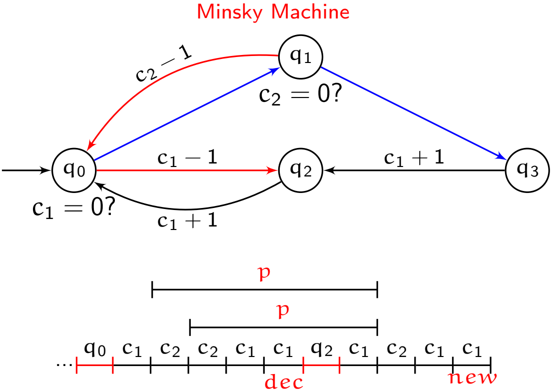
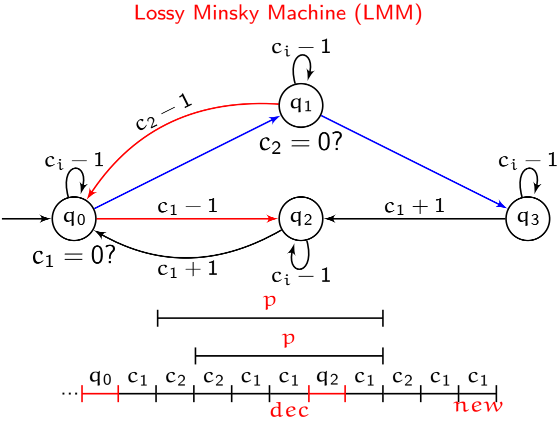
Complexity, Decidability and Undecidability



$$\bigvee_i c_i \land \neg new \to \langle \bar{A} \rangle p$$

The crucial part of the encoding is maintaining the correct value of the counters between two consecutive configurations.
In order to encode exactly the computation of a Minsky Machine p must represent a bijection. With the $\langle \bar{A} \rangle$ operator we can only garantee that p represents a surjective function.

Complexity, Decidability and Undecidability



$\bigvee_i c_i \wedge \neg new \to \langle \bar{A} \rangle p$

The crucial part of the encoding is maintaining the correct value of the counters between two consecutive configurations.
In order to encode exactly the computation of a Minsky Machine p must represent a bijection. With the $\langle \bar{A} \rangle$ operator we can only garantee that p represents a surjective function.

For every computation of a Lossy Minsky Machine there exists a model of this formula which encodes it, and vice versa.

Complexity, Decidability and Undecidability

In particular:

- we can reduce the problem of Reachability for an LMM (not primitive recursive) to the satisfiability problem of an $\mathrm{A}\bar{\mathrm{A}}\mathrm{B}\bar{\mathrm{B}}$ formula over finite linear orders;

Complexity, Decidability and Undecidability

In particular:

- we can reduce the problem of Reachability for an LMM (not primitive recursive) to the satisfiability problem of an $A\bar{A}B\bar{B}$ formula over finite linear orders;

- by adding to $A\bar{A}B\bar{B}$ any other modality among $\langle E \rangle$, $\langle O \rangle$, $\langle D \rangle$ and their transposes we can encode the Reachability problem for Minsky Machines (undecidable).

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  $\mathrm{A}\bar{\mathrm{A}}\mathrm{B}\bar{\mathrm{B}}$ turns out to be maximal with respect to the decidability over the class of finite linear orders.

Complexity, Decidability and Undecidability

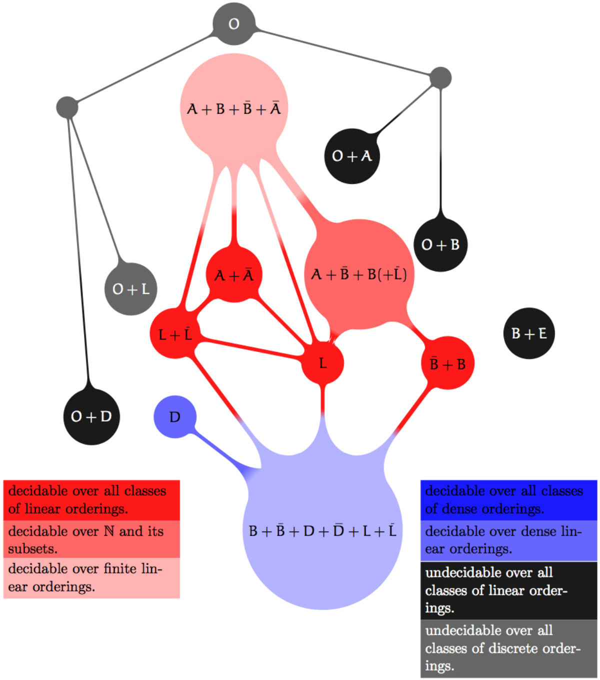
In particular:

- we can reduce the problem of Reachability for an LMM (not primitive recursive) to the satisfiability problem of an $A\bar{A}B\bar{B}$ formula over finite linear orders;

- by adding to $A\bar{A}B\bar{B}$ any other modality among $\langle E \rangle, \langle O \rangle, \langle D \rangle$ and their transposes we can encode the Reachability problem for Minsky Machines (undecidable).

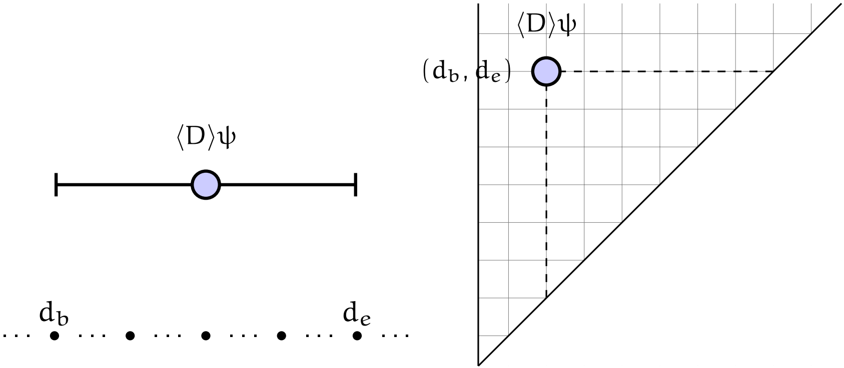
  $A\bar{A}B\bar{B}$ turns out to be maximal with respect to the decidability over the class of finite linear orders.

- Moreover, we can reduce the problem of Structural Termination (undecidable) for an LMM to the satisfiability problem for $A\bar{A}B\bar{B}$ over linear orders which feature at least one infinite ascending sequence;


O
A + B + B̄ + Ā
O + A
O + B
A + Ā
A + B̄ + B(+L̄)
O + L
L + L̄
L
B̄ + B
B + E
O + D
D
B + B̄ + D + D̄ + L + L̄
decidable over all classes of linear orderings.
decidable over ℕ and its subsets.
decidable over finite linear orderings.
decidable over all classes of dense orderings.
decidable over dense linear orderings.
undecidable over all classes of linear orderings.
undecidable over all classes of discrete orderings.

Open Problem

Decidability of the logic D over the finite linear orders.



Open Problem

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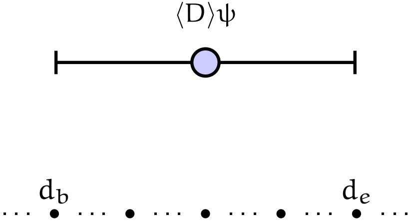

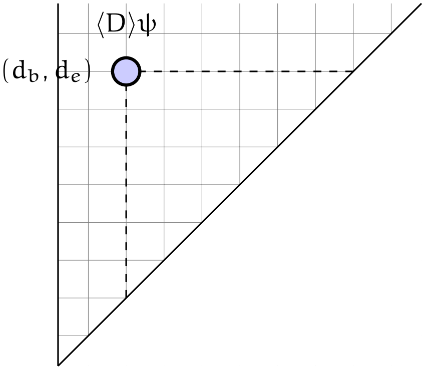

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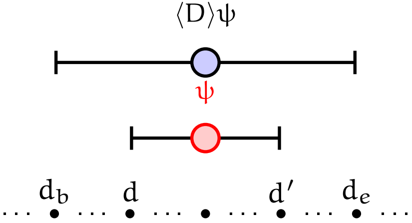

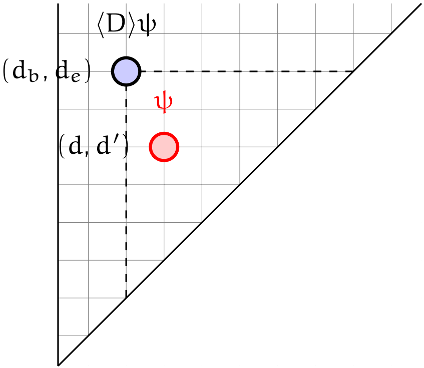

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