Structures of Diversity

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ICTCS 2010

Zero–Error Capacity

Shannon 1956*

Suppose we want to transmit messages across a channel (where some symbols may be distorted) to a receiver: What is the maximum rate of transmission such that the receiver may recover the original message without errors?

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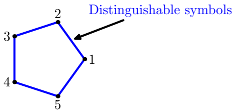

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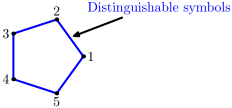

- Transmission Rate: The maximum number of bits that can be transmitted without errors per channel use.

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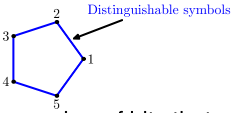

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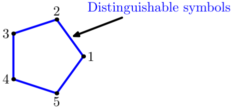

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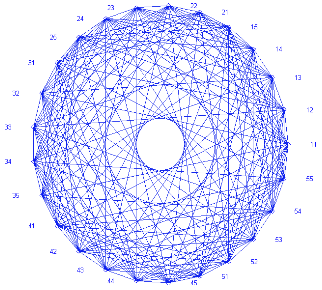

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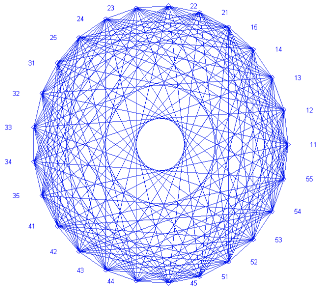

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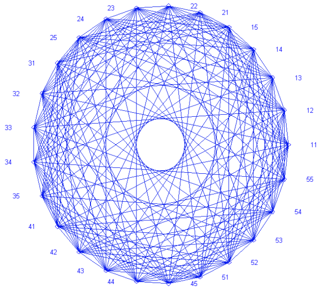

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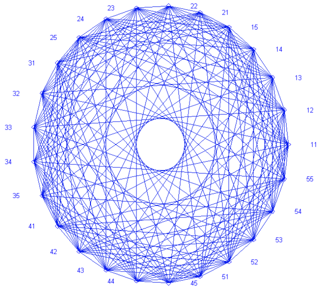

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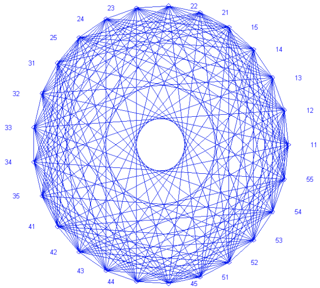

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C a clique in G^n then

$$\mathbf{x}, \mathbf{y} \in C \Rightarrow \exists i \in [n], \{x_i, y_i\} \in E(G).$$

Definition

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$$C(G) = \lim_{n \to +\infty} \frac{1}{n} \log \omega(G^n)$$

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Determining the value of $C(C_7)$ is still open!

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Generalizations

- Graphs [Sha56]
- Directed Graphs [KS92, GKV92]
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Connections

Extremal Combinatorics

- Perfect Graphs [Ber62]
- Qualitative Independence [Rén71, GKV93]

Information Theory

- Perfect hashing [FK84]
- Zero error list decoding [Eli57]
- Zero error capacity of compound channels [BBT59, Dob59, Wol60, NR05]

Generalization to Infinite Graphs ?

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$$\Downarrow$$

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Permutations

G–different Permutations.

[KSS09] J. Körner, G. Simonyi and B.Sinaimeri, On types of growth for graph–different permutations,*J. Combin. Theory Ser. A*, **116**, 713–723, 2009.

$$\pi = \pi(1)\pi(2)\dots\pi(n)$$

Definition

G an infinite graph with $V(G) = \mathbb{N}$. Two permutations π, ρ of $[n]$ are said G–different if $\exists i \in [n]$ such that $\{\pi(i), \rho(i)\} \in E(G)$.

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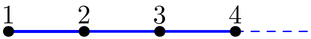

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- $\pi = 12345$
  $\rho = 13245$

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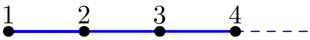

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  Not G–different.

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Conjecture

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Surprisingly for the complement graph of* L *an exact formula is found ...

Theorem

Let $\overline{L}$ be the complement graph of semi-infinite path, then

$$T(n, \overline{L}) = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}$$

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- $|C(\pi)| = 2^{\lfloor \frac{n}{2} \rfloor}$
- If π, ρ are $\overline{L}$–different then $C(\pi) \cap C(\rho) = \emptyset$.

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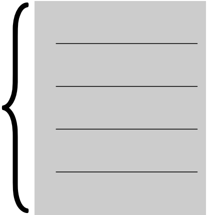

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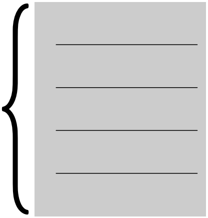
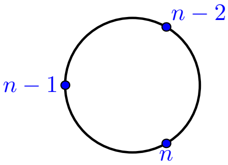


Insert $n-1$ and n



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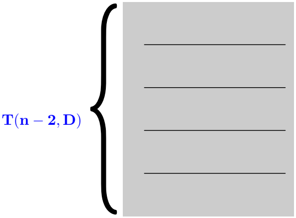

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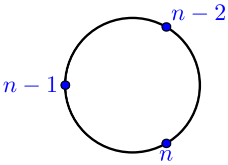

$$T(n, \overline{L}) \geq \binom{n}{2} T(n-2, \overline{L})$$

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Study the relations between $T(n,G)$ ***and*** $T(n,\overline{G})$

Study the asymptotic of $\dfrac{T(n,F)\,T(n,G)}{T(n,F \cup G)}$

The Shannon zero–error capacity is a special case of the problem of determining the asymptotic growth of $T(n, G)$.

Example

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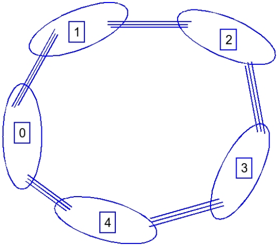

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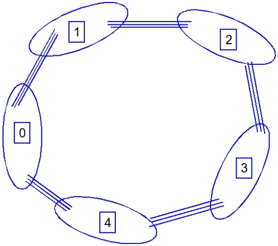

$$\lim_{n \to +\infty} \frac{1}{n} \log T(n, G) = C(C_5)$$

Difference and Similarity

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- Erdős–Ko–Rado [EKR61]
- Intersection theorems for permutations [EFP]

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Forbiddance Problems

[FKMS] Z. Füredi, I. Kantor, A. Monti and B. Sinaimeri, On sets of pairwise reverse free ordered triples, SIAM J. Discrete Math. Volume 24, Issue 3, pp. 964-978 (2010)

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For G, denote by $\vec{\mathcal{G}}(G)$ the family of all the orientations of G.

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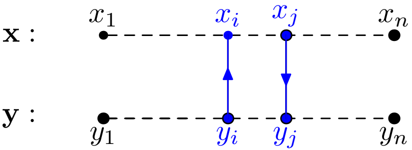

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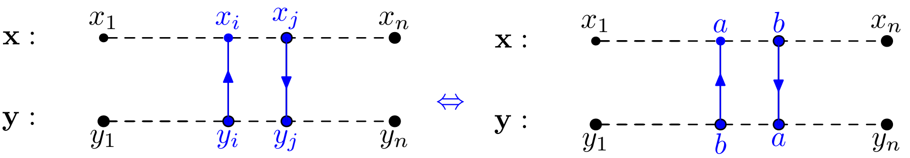

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Partial permutations, i.e. ordered sets of k-elements.

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Case $\mathbf{k} = \mathbf{3}$

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Theorem

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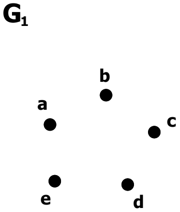

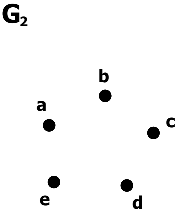

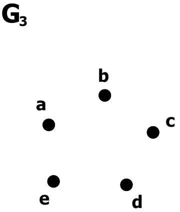

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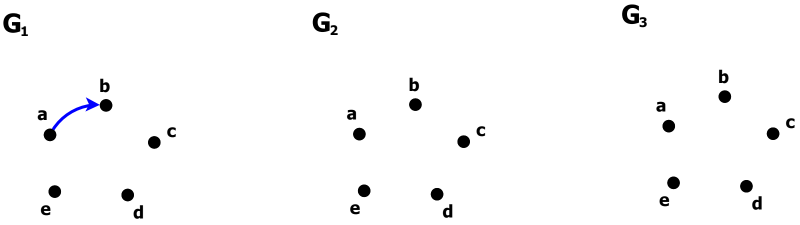

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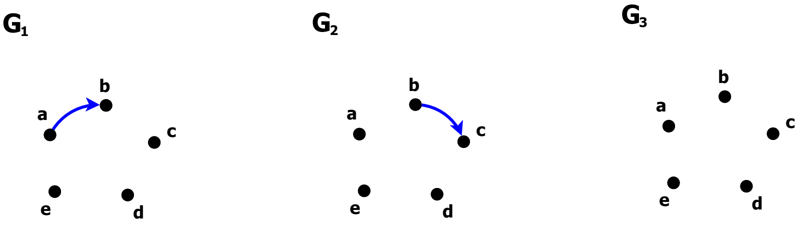

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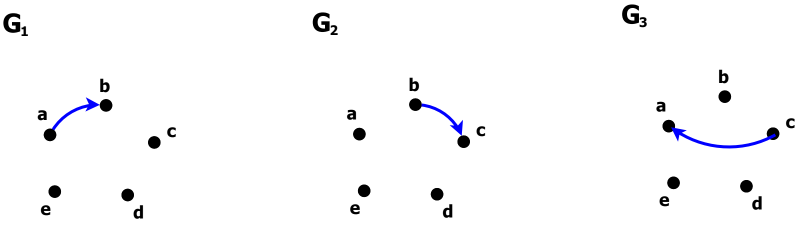

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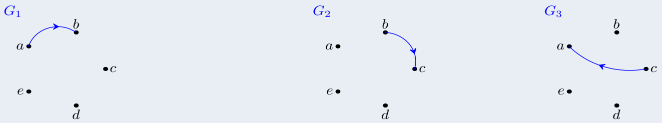

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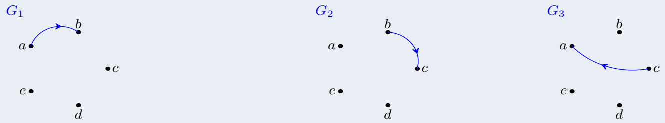

D with $\mathbf{E(D) = E(G_1) \cap E(G_2) \cap E(G_3)}$

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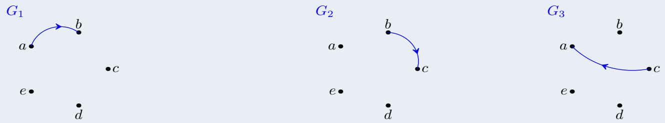

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a	**b**	**c**
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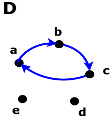
c	**a**	**b**

$\Longrightarrow$



Cancellative Families

[KS07] J. Körner and B.Sinaimeri, On cancellative set families, *Combinatorics, Prob. Computing*, **16**, 767–773, 2007.

Capacity of a k-uniform hypergraph

G–difference: from binary relations to relations involving k-sets of strings

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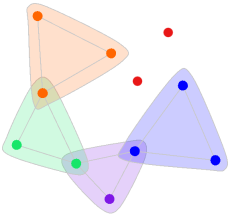

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Problem [KS88]

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Open Problems!!

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Extremal Combinatorics $\Leftrightarrow$ Information Theory

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