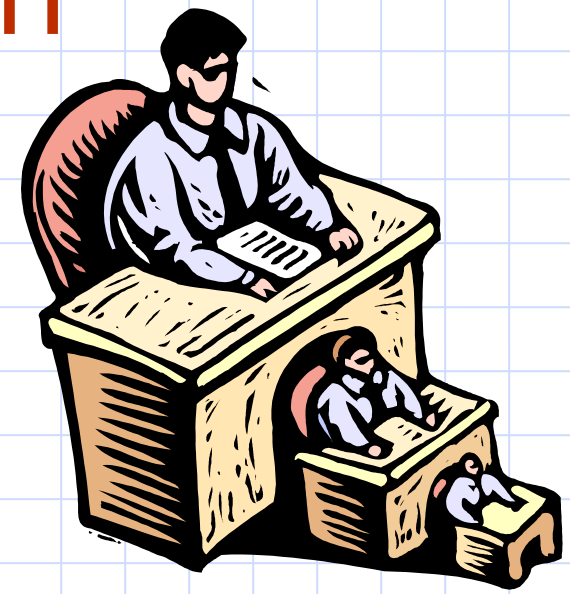
Using Recursion



The Recursion Pattern

- **Recursion**: when a method calls itself
- Classic example: the factorial function:
  - n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a Java method:

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
     if  (n=0)  return  1; // basis case
     return  n * recursiveFactorial(n-1);//
}
```

Linear Recursion

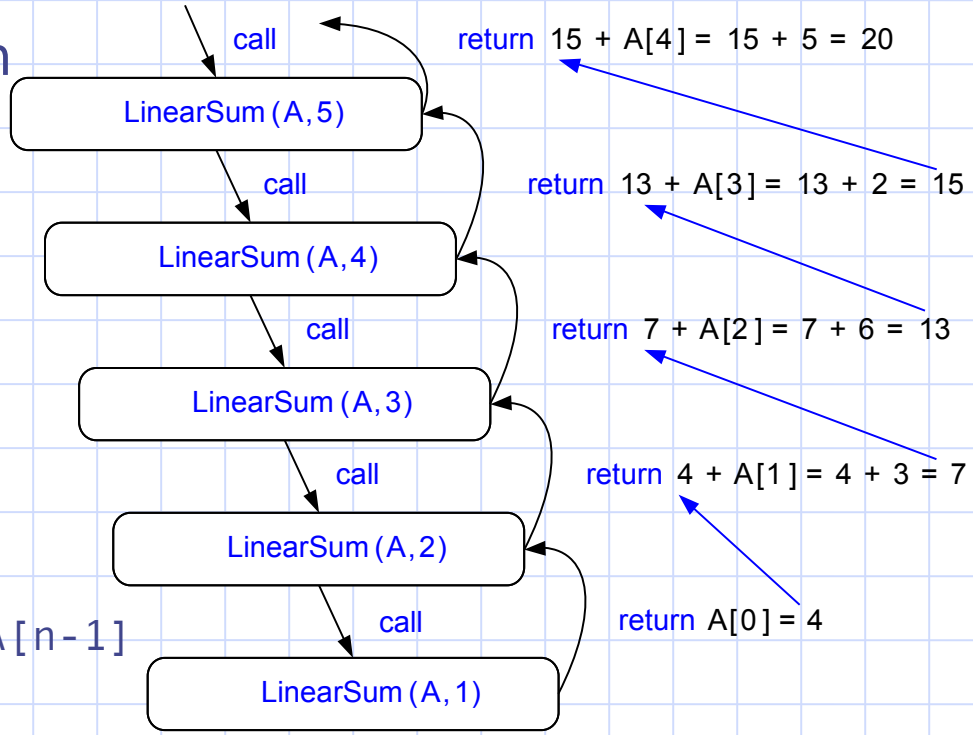
- Test for base cases
  - Begin by testing for a ***set of base cases*** (there should be at least one).
  - Every possible chain of recursive calls must reach a base case
- Recur once
  - Perform a ***single recursive call***
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that ***it makes progress towards*** a base case.

Example of Linear Recursion

LinearSum(*A*, *n*):
Input:
An integer array *A* and an integer *n* such that *A* has at least *n* elements
Output:
The sum of the first *n* integers in *A*

```
if (n == 1) then
  return A[0]
else
return LinearSum(A,n-1)+ A[n-1]
```

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(*A*, *i*, *j*):

Input: An array *A* and nonnegative integer indices *i* and *j*

Output: The reversal of the elements in *A* starting at index *i* and ending at *j*

```
if i < j then
  temp = A[i]     // Swap A[i] and A[j]
  A[i] = A[j]
  A[j] = temp
  ReverseArray(A, i+1, j-1)
  return
```

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n)=\begin{cases} 1 & \text{if } n=0 \\ x \cdot p(x,n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in ***O(n)*** time (and make **n** recursive calls).
- We can do better than this

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

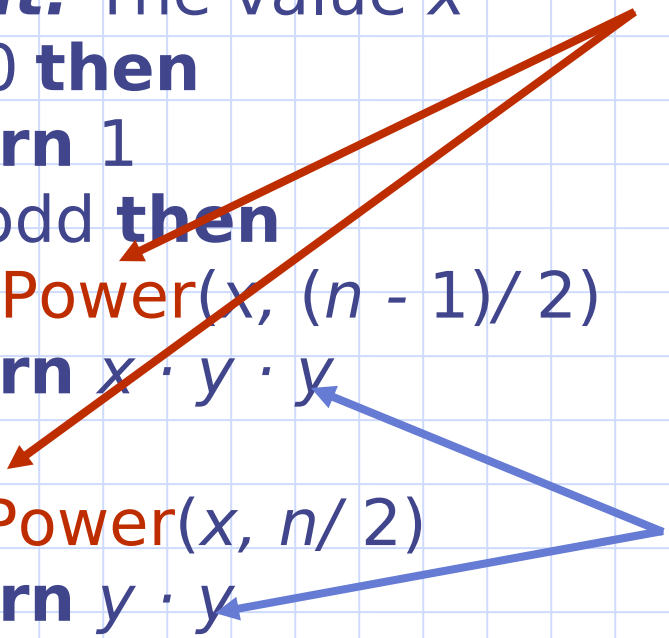
$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$
$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$
$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$
$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$$

Recursive Squaring Method

Algorithm Power(x, n):
    Input: A number x and integer $n = 0$
    Output: The value x^n
    if $n = 0$ **then**
        return 1
    if n is odd **then**
        y = Power(x, (n - 1)/ 2)
        return $x \cdot y \cdot y$
    else
        y = Power(x, n/ 2)
        return $y \cdot y$

Analysis

Algorithm Power(x, n):
    Input: A number x and integer $n = 0$
    Output: The value x^n
    if $n = 0$ **then**
        return 1
    if n is odd **then**
        $y = \text{Power}(x, (n - 1)/2)$
        return $x \cdot y \cdot y$
    else
        $y = \text{Power}(x, n/2)$
        return $y \cdot y$



The recurrence that describes the complexity of this algorithm is $T(n) = T(n/2) + c$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in **O(log n)** time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step (an example: the array reversal method).
- Such methods can be easily converted to non-recursive methods (which saves on some resources).

Algorithm IterativeReverseArray(*A*, *i*, *j*):
Input: An array *A* and nonnegative integer indices *i* and *j*
Output: The reversal of the elements in *A* starting at index *i* and ending at *j*
while *i* < *j* **do**
Swap *A*[*i*] and *A*[*j*]
i = *i* + 1
j = *j* - 1
return

Binary Recursion

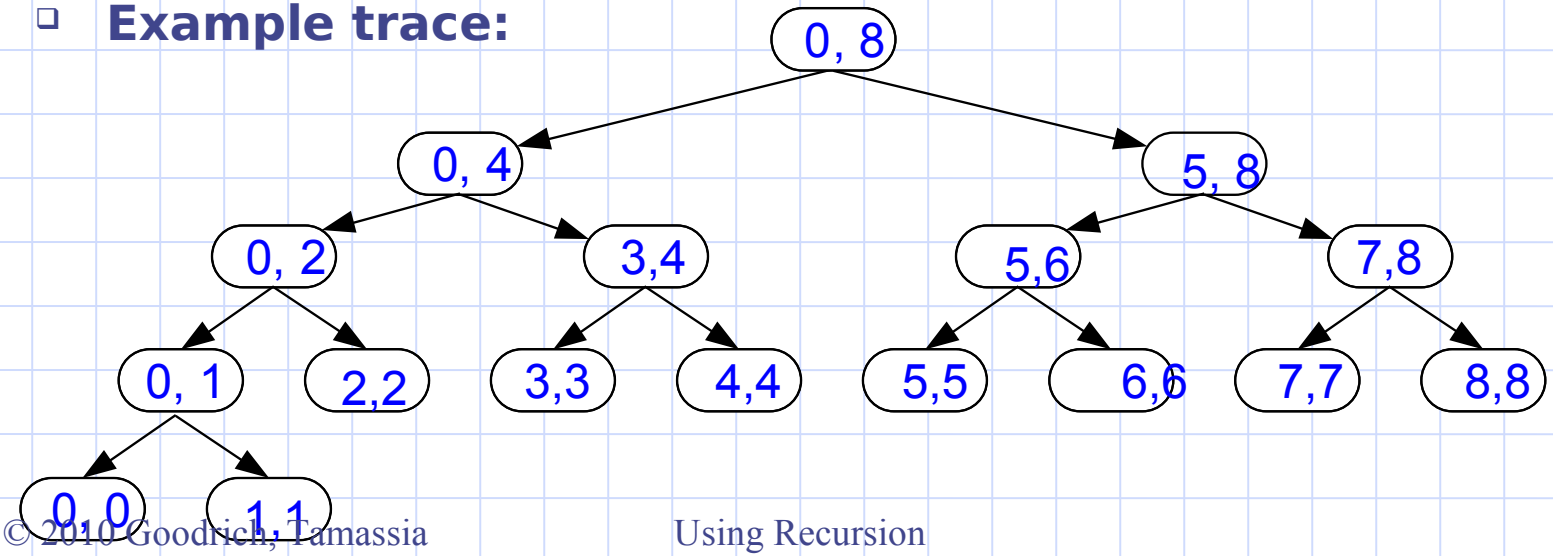
- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.

A Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(*A, i, j*):
 Input: An array *A, two* indexes *i* and *j*
 Output: The sum of the integers in *A between i and j*
 if (i=j) **then**
 return *A*[*i*]
 return BinarySum(*A, i, (i+j)/* 2) + BinarySum(*A, (i+j)/2 +1, j*)

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$F_0 = 0$

$F_1 = 1$

$F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):

Input: Non-negative integer k

Output: The kth Fibonacci number F_k

if $k <= 1$ **return** k

returnn BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1, n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

Algorithm LinearFibonacci(k):
Input: A non-negative integer k
Output: Pair of Fibonacci numbers (F_k , F_{k-1})
if $k = 1$ **then return** (1, 0)
(i, j) = LinearFibonacci($k - 1$)
return (i + j, i)

- LinearFibonacci makes k–1 recursive calls

An iterative Fibonacci Algorithm

- **Algorithm** IterativeFibonacci(k):
  Input: A non-negative integer k
  Output: The k-th Fibonacci number

```
int A[] 0 = new int[k+1];
A[0]=0; A[1] = 1;
for (int i=2; i<=k; i++)
        A[i] = A[i-1] + A[i-2]
return A[k]
```